Problem 4 (15%)

Consider a directed graph $G = (V, E)$ where $|V| = n$ and $|E| = n^2/2$.

Question 4.1

What is the runtime of Dijkstra's single-source shortest path algorithm on the graph G when the priority queue is implemented using a linked list, a binary heap, and a Fibonacci heap? Which one of these three implementations is preferable?

Solution: With linked lists: $O(n^2)$. With binary heap: $O(n^2 \lg n)$. With Fibonacci heap: $O(n^2)$. Clearly, the linked list implementation is preferable since it is much simpler than the Fibonacci heap implementation.

A library decides to store its books according to their heights in order to minimize storage costs.

Suppose the book heights are $H_1, H_2, \ldots, H_n$ with $H_1 < H_2 < \cdots < H_n$ and let L_i be the shelf length required to hold all books of height H_i. (Notice that n is the number of different heights, *not* the number of books.)

The cost of x_i centimeters of shelving with height H_i is $F_i + C_i x_i$ where F_i is a fixed cost independent on the length and C_i is the cost of the shelf per centimeter. Notice that the fixed cost F_i can be saved by not buying shelves of every possible height since a shelf of height H_i can be used to store books of smaller height.

Question 4.2

We want to determine the length of shelving of each height to minimize the total cost of shelving. Show how to formulate this problem as a shortest path problem.

Solution: We construct a complete directed weighted graph $G = (V, E)$ with vertices $V = \{0, 1, \ldots, n\}$ and edge weights given by

$$w(i, j) = F_j + C_j \left(\sum_{k=i+1}^{j} L_k \right) \quad \text{for } 0 \leq i < j \leq n.$$

The weight of shortest path from vertex 0 to vertex n is the minimal total cost. The vertices on a shortest path from vertex 0 to vertex n indicate the optimal shelve heights (e.g., if the shortest path is $\langle 0, 3, 7, \ldots \rangle$, one should buy $(L_1 + L_2 + L_3)$ centimeters of shelves with height H_3 and $(L_4 + L_5 + L_6 + L_7)$ centimeters of shelves with height H_7).