Quantified Boolean Formulas

Note for IAIP Fall 2005
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Quantified Boolean Formulas (QBF) [1] provides a concise notation for complex operations on Boolean formulas which we will use extensively to define BDD operations. QBF is ordinary propositional logic extended with quantification of Boolean variables.

**Definition 1 (QBF syntax)** Given a set $V = \{v_1, \cdots, v_n\}$ of propositional variables, $QBF(V)$ formulas are inductively defined by

- every variable in $V$ is a formula,
- if $f$ and $g$ are formulas, then so are $\neg f$, $f \land g$, and $f \lor g$, and
- if $f$ is a formula and $v \in V$, then $\exists v \cdot f$ and $\forall v \cdot f$ are formulas.

A truth assignment for $QBF(V)$ is a function $\sigma : V \to \mathbb{B}$. We will use the notation $\sigma(v \leftarrow a)$ for the truth assignment defined by

$$
\sigma(v \leftarrow a)(w) = \begin{cases} 
  a & \text{if } v = w \\
  \sigma(w) & \text{otherwise}.
\end{cases}
$$

**Definition 2 (QBF Semantics)** If $f$ is a formula in $QBF(V)$ and $\sigma$ is a truth assignment, we will write $\sigma \models f$ to denote that $f$ is true under the assignment $\sigma$. The relation $\models$ is defined inductively in the obvious manner

- $\sigma \models v$ iff $\sigma(v) = \text{true}$,
- $\sigma \models \neg f$ iff $\sigma \not\models f$,
- $\sigma \models f \lor g$ iff $\sigma \models f$ or $\sigma \models g$,
- $\sigma \models f \land g$ iff $\sigma \models f$ and $\sigma \models g$,
- $\sigma \models \exists v \cdot f$ iff $\sigma(v \leftarrow \text{false}) \models f$ or $\sigma(v \leftarrow \text{true}) \models f$,
- $\sigma \models \forall v \cdot f$ iff $\sigma(v \leftarrow \text{false}) \models f$ and $\sigma(v \leftarrow \text{true}) \models f$. 

For a vector $\vec{v} = (v_1, \cdots, v_m)$ of propositional variables in $V$, we define the abbreviations

$$
\exists \vec{v}. f \equiv \exists v_1. (\cdots (\exists v_m. f) \cdots)
$$

(1)

$$
\forall \vec{v}. f \equiv \forall v_1. (\cdots (\forall v_m. f) \cdots).
$$

(2)

The *support* of a formula $f$ is the set of variables that $f$ depends on $\{v \in V \mid f|_{v\leftarrow \text{true}} \neq f|_{v\leftarrow \text{false}}\}$.

**References**