# Tutorial 3

# Using MATLAB in Linear Algebra

## Math 221

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One of the nice features of MATLAB is its ease of computations with vectors and matrices. In this tutorial the following topics are discussed: vectors and matrices in MATLAB, solving systems of linear equations, the inverse of a matrix, determinants, vectors in n-dimensional Euclidean space, linear transformations, real vector spaces and the matrix eigenvalue problem. Applications of linear algebra to the curve fitting, message coding and computer graphics are also included.

## 3.1 Special characters and MATLAB functions used in Tutorial 3

For the reader's convenience we include lists of special characters and MATLAB functions that are used in this tutorial.

| Special characters | |
|---|---|
| **;** | Semicolon operator |
| **'** | Conjugated transpose |
| **.'** | Transpose |
| ***** | Times |
| **.** | Dot operator |
| **^** | Power operator |
| **[ ]** | Emty vector operator |
| **:** | Colon operator |
| **=** | Assignment |
| **==** | Equality |
| **** | Backslash or left division |
| **/** | Right division |
| **i, j** | Imaginary unit |
| **~** | Logical not |
| **~=** | Logical not equal |
| **&** | Logical and |
| **\|** | Logical or |
| **{ }** | Cell |

| Function | Description |
|---|---|
| **acos** | Inverse cosine |
| **axis** | Control axis scaling and appearance |
| **char** | Create character array |
| **chol** | Cholesky factorization |
| **cos** | Cosine function |
| **cross** | Vector cross product |
| **det** | Determinant |
| **diag** | Diagonal matrices and diagonals of a matrix |
| **double** | Convert to double precision |
| **eig** | Eigenvalues and eigenvectors |
| **eye** | Identity matrix |
| **fill** | Filled 2-D polygons |
| **fix** | Round towards zero |
| **fliplr** | Flip matrix in left/right direction |
| **flops** | Floating point operation count |
| **grid** | Grid lines |
| **hadamard** | Hadamard matrix |
| **hilb** | Hilbert matrix |
| **hold** | Hold current graph |
| **inv** | Matrix inverse |
| **isempty** | True for empty matrix |
| **legend** | Graph legend |
| **length** | Length of vector |
| **linspace** | Linearly spaced vector |
| **logical** | Convert numerical values to logical |
| **magic** | Magic square |
| **max** | Largest component |
| **min** | Smallest component |
| **norm** | Matrix or vector norm |
| **null** | Null space |
| **num2cell** | Convert numeric array into cell array |
| **num2str** | Convert number to string |
| **ones** | Ones array |
| **pascal** | Pascal matrix |
| **plot** | Linear plot |
| **poly** | Convert roots to polynomial |
| **polyval** | Evaluate polynomial |
| **rand** | Uniformly distributed random numbers |
| **randn** | Normally distributed random numbers |
| **rank** | Matrix rank |
| **reff** | Reduced row echelon form |
| **rem** | Remainder after division |
| **reshape** | Change size |
| **roots** | Find polynomial roots |
| **sin** | Sine function |
| **size** | Size of matrix |
| **sort** | Sort in ascending order |

| | |
|---|---|
| **subs** | Symbolic substitution |
| **sym** | Construct symbolic bumbers and variables |
| **tic** | Start a stopwatch timer |
| **title** | Graph title |
| **toc** | Read the stopwatch timer |
| **toeplitz** | Tioeplitz matrix |
| **tril** | Extract lower triangular part |
| **triu** | Extract upper triangular part |
| **vander** | Vandermonde matrix |
| **varargin** | Variable length input argument list |
| **zeros** | Zeros array |

## 3.2 Vectors and matrices in MATLAB

The purpose of this section is to demonstrate how to create and transform vectors and matrices in MATLAB.

This command creates a row vector

```
a = [1 2 3]

a =
     1     2     3
```

Column vectors are inputted in a similar way, however, semicolons must separate the components of a vector

```
b = [1;2;3]

b =
     1
     2
     3
```

The *quote operator* **'** is used to create the *conjugate transpose* of a vector (matrix) while the *dot-quote operator* **.'** creates the *transpose* vector (matrix). To illustrate this let us form a complex vector **a + i*b'** and next apply these operations to the resulting vector to obtain

```
(a+i*b')'

ans =
   1.0000 - 1.0000i
   2.0000 - 2.0000i
   3.0000 - 3.0000i
```

while

```
(a+i*b').'

ans =
   1.0000 + 1.0000i
   2.0000 + 2.0000i
   3.0000 + 3.0000i
```

Command **length** returns the number of components of a vector

```
length(a)

ans =
     3
```

The *dot operator*. plays a specific role in MATLAB. It is used for the componentwise applicat of the operator that follows the dot operator

```
a.*a

ans =
     1     4     9
```

The same result is obtained by applying the *power operator* **^** to the vector **a**

```
a.^2

ans =
     1     4     9
```

Componentwise division of vectors **a** and **b** can be accomplished by using the *backslash opera* **** together with the dot operator **.**

```
a.\b'

ans =
     1     1     1
```

For the purpose of the next example let us change vector **a** to the column vector

```
a = a'

a =
     1
     2
     3
```

The *dot product* and the *outer product* of vectors **a** and **b** are calculated as follows

```
dotprod = a'*b
```

```
dotprod =
    14
outprod = a*b'

outprod =
     1     2     3
     2     4     6
     3     6     9
```

The *cross product* of two three-dimensional vectors is calculated using command **cross**. Let the vector **a** be the same as above and let

```
b = [-2 1 2];
```

Note that the semicolon after a command avoids display of the result. The cross product of **a** and **b** is

```
cp = cross(a,b)

cp =
     1    -8     5
```

The cross product vector **cp** is perpendicular to both **a** and **b**

```
[cp*a cp*b']

ans =
     0     0
```

We will now deal with operations on matrices. Addition, subtraction, and scalar multiplication are defined in the same way as for the vectors.

This creates a 3-by-3 matrix

```
A = [1 2 3;4 5 6;7 8 10]

A =
     1     2     3
     4     5     6
     7     8    10
```

Note that the *semicolon operator* **;** separates the rows. To extract a submatrix **B** consisting of rows 1 and 3 and columns 1 and 2 of the matrix **A** do the following

```
B = A([1 3], [1 2])

B =
     1     2
     7     8
```

To interchange rows 1 and 3 of **A** use the vector of row indices together with the colon operator

```
C = A([3 2 1],:)

C =
     7     8    10
     4     5     6
     1     2     3
```

The *colon operator* **:** stands for *all columns* or *all rows*. For the matrix **A** from the last example the following command

```
A(:)

ans =
     1
     4
     7
     2
     5
     8
     3
     6
    10
```

creates a vector version of the matrix **A**. We will use this operator on several occasions.

To delete a row (column) use the *empty vector operator* **[ ]**

```
A(:, 2) = []

A =
     1     3
     4     6
     7    10
```

Second column of the matrix **A** is now deleted. To insert a row (column) we use the technique creating matrices and vectors

```
A = [A(:,1) [2 5 8]' A(:,2)]
A =
     1     2     3
     4     5     6
     7     8    10
```

Matrix **A** is now restored to its original form.

Using MATLAB commands one can easily extract those entries of a matrix that satisfy an imp condition. Suppose that one wants to extract all entries of that are greater than one. First, we define a new matrix **A**

```
A = [-1 2 3;0 5 1]

A =
    -1     2     3
     0     5     1
```

Command **A > 1** creates a matrix of zeros and ones

```
A > 1

ans =
     0     1     1
     0     1     0
```

with ones on these positions where the entries of **A** satisfy the imposed condition and zeros everywhere else. This illustrates *logical addressing* in MATLAB. To extract those entries of the matrix **A** that are greater than one we execute the following command

```
A(A > 1)

ans =
     2
     5
     3
```

The dot operator **.** works for matrices too. Let now

```
A = [1 2 3; 3 2 1] ;
```

The following command

```
A.*A

ans =
     1     4     9
     9     4     1
```

computes the entry-by-entry product of **A** with **A**. However, the following command

```
A*A

¨??? Error using ==> *
Inner matrix dimensions must agree.
```

generates an error message.

Function **diag** will be used on several occasions. This creates a *diagonal matrix* with the diagonal entries stored in the vector **d**

```
d = [1 2 3];

D = diag(d)

D =
     1     0     0
     0     2     0
     0     0     3
```

To extract the main diagonal of the matrix **D** we use function **diag** again to obtain

```
d = diag(D)

d =
     1
     2
     3
```

What is the result of executing of the following command?

```
diag(diag(d));
```

In some problems that arise in linear algebra one needs to calculate a *linear combination* of several matrices of the same dimension. In order to obtain the desired combination both the coefficients and the matrices must be stored in *cells.* In MATLAB a cell is inputted using curly braces**{ }**. This

```
c = {1,-2,3}

c =
    [1]    [-2]    [3]
```

is an example of the cell. Function **lincomb** will be used later on in this tutorial.

```
function M = lincomb(v,A)

% Linear combination M of several matrices of the same size.
% Coefficients v = {v1,v2,…,vm} of the linear combination and the
% matrices A = {A1,A2,...,Am} must be inputted as cells.

m = length(v);
[k, l] = size(A{1});
M = zeros(k, l);
for i = 1:m
   M = M + v{i}*A{i};
end
```

## 3.3 Solving systems of linear equations

MATLAB has several tool needed for computing a solution of the system of linear equations.

Let **A** be an m-by-n matrix and let **b** be an m-dimensional (column) vector. To solve the linear system **Ax = b** one can use the *backslash operator* **** , which is also called the *left division*.

**1. Case m = n**

In this case MATLAB calculates the exact solution (modulo the roundoff errors) to the system in question.

Let

```
A = [1 2 3;4 5 6;7 8 10]

A =
     1     2     3
     4     5     6
     7     8    10
```

and let

```
b = ones(3,1);
```

Then

```
x = A\b

x =
   -1.0000
    1.0000
    0.0000
```

In order to verify correctness of the computed solution let us compute the *residual vector* **r**

```
r = b - A*x

r =
  1.0e-015 *
    0.1110
    0.6661
    0.2220
```

Entries of the computed residual **r** theoretically should all be equal to zero. This example illustrates an effect of the roundoff erros on the computed solution.

**2. Case m > n**

If **m > n**, then the system **Ax = b** is *overdetermined* and in most cases system is inconsistent. A solution to the system **Ax = b**, obtained with the aid of the backslash operator **** , is the *least-squares solution.*

Let now

```
A = [2 –1; 1 10; 1 2];
```

and let the vector of the right-hand sides will be the same as the one in the last example. Then

```
x = A\b
x =
    0.5849
    0.0491
```

The residual **r** of the computed solution is equal to

```
r = b - A*x

r =
   -0.1208
   -0.0755
    0.3170
```

Theoretically the residual **r** is orthogonal to the *column space* of **A**. We have

```
r'*A

ans =
  1.0e-014 *
    0.1110
    0.6994
```

**3. Case m < n**

If the number of unknowns exceeds the number of equations, then the linear system is *underdetermined*. In this case MATLAB computes a *particular solution* provided the system is consistent. Let now

```
A = [1 2 3; 4 5 6];
b = ones(2,1);
```

Then

```
x = A\b

x =
   -0.5000
         0
    0.5000
```

A *general solution* to the given system is obtained by forming a linear combination of **x** with th columns of the *null space* of **A**. The latter is computed using MATLAB function **null**

```
z = null(A)

z =
    0.4082
   -0.8165
    0.4082
```

Suppose that one wants to compute a solution being a linear combination of **x** and **z**, with coefficients **1** and **–1**. Using function **lincomb** we obtain

```
w = lincomb({1,-1},{x,z})

w =
   -0.9082
    0.8165
    0.0918
```

The residual **r** is calculated in a usual way

```
r = b - A*w

r =
  1.0e-015 *
   -0.4441
    0.1110
```

## 3.4 Function rref and its applications

The built-in function **rref** allows a user to solve several problems of linear algebra. In this section we shall employ this function to compute a solution to the system of linear equations and also to find the rank of a matrix. Other applications are discussed in the subsequent sections of this tutorial.

Function **rref** takes a matrix and returns the *reduced row echelon form* of its argument. Syntax of the **rref** command is

**B = rref(A)** or **[B, pivot] = rref(A)**

The second output parameter **pivot** holds the indices of the pivot columns.

Let

```
A = magic(3); b = ones(3,1);
```

A solution **x** to the linear system **Ax = b** is obtained in two steps. First the augmented matrix of the system is transformed to the *reduced echelon form* and next its last column is extracted

```
[x, pivot] = rref([A b])

x =
    1.0000         0         0    0.0667
         0    1.0000         0    0.0667
         0         0    1.0000    0.0667
pivot =
     1     2     3
```

```
x = x(:,4)

x =
    0.0667
    0.0667
    0.0667
```

The residual of the computed solution is

```
b - A*x

ans =
     0
     0
     0
```

Information stored in the output parameter **pivot** can be used to compute the rank of the matrix

```
length(pivot)

ans =
     3
```

## 3.5 The inverse of a matrix

MATLAB function **inv** is used to compute the inverse matrix.

Let the matrix **A** be defined as follows

```
A = [1 2 3;4 5 6;7 8 10]

A =
     1     2     3
     4     5     6
     7     8    10
```

Then

```
B = inv(A)

B =
   -0.6667   -1.3333    1.0000
   -0.6667    3.6667   -2.0000
    1.0000   -2.0000    1.0000
```

In order to verify that B is the inverse matrix of A it sufficies to show that **A*B = I** and **B*A = I**, where **I** is the 3-by-3 identity matrix. We have

```
A*B

ans =
    1.0000         0   -0.0000
         0    1.0000         0
         0         0    1.0000
```

In a similar way one can check that **B*A = I**.

The *Pascal matrix*, named in MATLAB **pascal**, has several interesting properties. Let

```
A = pascal(3)

A =
     1     1     1
     1     2     3
     1     3     6
```

Its inverse **B**

```
B = inv(A)

B =
     3    -3     1
    -3     5    -2
     1    -2     1
```

is the matrix of integers. The *Cholesky triangle* of the matrix **A** is

```
S = chol(A)

S =
     1     1     1
     0     1     2
     0     0     1
```

Note that the upper triangular part of **S** holds the binomial coefficients. One can verify easily that **A** = **S'*S**.

Function **rref** can also be used to compute the inverse matrix. Let **A** is the same as above. We create first the augmented matrix **B** with **A** being followed by the identity matrix of the same size as **A**. Running function **rref** on the augmented matrix and next extracting columns four through six of the resulting matrix, we obtain

```
B = rref([A eye(size(A))]);

B = B(:, 4:6)

B =
     3    -3     1
    -3     5    -2
     1    -2     1
```

To verify this result, we compute first the product **A** ***B**

```
A*B

ans =
     1     0     0
     0     1     0
     0     0     1
```

and next **B***A**

```
B*A

ans =
     1     0     0
     0     1     0
     0     0     1
```

This shows that **B** is indeed the inverse matrix of **A**.

## 3.6 Determinants

In some applications of linear algebra knowledge of the determinant of a matrix is required. MATLAB built-in function **det** is designed for computing determinants.

Let

```
A = magic(3);
```

Determinant of **A** is equal to

```
det(A)

ans =
  -360
```

One of the classical methods for computing determinants utilizes a *cofactor expansion*. For mo details, see e.g., [2], pp. 103-114.

Function **ckl = cofact(A, k, l)** computes the cofactor **ckl** of the **$a_{kl}$** entry of the matrix **A**

```
function ckl = cofact(A,k,l)

% Cofactor ckl of the a_kl entry of the matrix A.

[m,n] = size(A);
if m ~= n
   error('Matrix must be square')
```

```
end
B = A([1:k-1,k+1:n],[1:l-1,l+1:n]);
ckl = (-1)^(k+l)*det(B);
```

Function **d = mydet(A)** implements the method of cofactor expansion for computing determinants

```
function d = mydet(A)

% Determinant d of the matrix A. Function cofact must be
% in MATLAB's search path.

[m,n] = size(A);
if m ~= n
   error('Matrix must be square')
end
a = A(1,:);
c = [];
for l=1:n
   c1l = cofact(A,1,l);
   c = [c;c1l];
end
d = a*c;
```

Let us note that function **mydet** uses the cofactor expansion along the row **1** of the matrix **A**. Method of cofactors has a high computational complexity. Therefore it is not recommended for computations with large matrices. Its is included here for pedagogical reasons only. To measure a computational complexity of two functions **det** and **mydet** we will use MATLAB built-in function **flops**. It counts the number of *floating-point operations* (additions, subtractions, multiplications and divisions). Let

```
A = rand(25);
```

be a 25-by-25 matrix of uniformly distributed random numbers in the interval **( 0, 1 )**. Using function **det** we obtain

```
flops(0)
det(A)

ans =
   -0.1867

flops

ans =
       10100
```

For comparison, a number of flops used by function **mydet** is

```
flops(0)
mydet(A)

ans =
   -0.1867

flops

ans =
      223350
```

The *adjoint matrix* **adj(A)** of the matrix **A** is also of interest in linear algebra (see, e.g., [2], p.108).

```
function B = adj(A)

% Adjoint matrix B of the square matrix A.

[m,n] = size(A);
if m ~= n
   error('Matrix must be square')
end
B = [];
for k = 1:n
   for l=1:n
      B = [B;cofact(A,k,l)];
   end
end
B = reshape(B,n,n);
```

The adjoint matrix and the inverse matrix satisfy the equation

$$A^{-1} = adj(A)/det(A)$$

(see [2], p.110 ). Due to the high computational complexity this formula is not recommended f computing the inverse matrix.

## 3.7 Vectors in $\mathbb{R}^n$

The *2-norm* (*Euclidean norm*) of a vector is computed in MATLAB using function **norm**.

Let

```
a = -2:2

a =
    -2    -1     0     1     2
```

The 2-norm of **a** is equal to

```
twon = norm(a)
```

```
twon =
    3.1623
```

With each nonzero vector one can associate a *unit vector* that is parallel to the given vector. For instance, for the vector **a** in the last example its unit vector is

```
unitv = a /twon

unitv =
   -0.6325   -0.3162         0    0.3162    0.6325
```

The angle $\theta$ between two vectors **a** and **b** of the same dimension is computed using the formula

$$\theta = \arccos(\mathbf{a.b}/||\mathbf{a}||\ ||\mathbf{b}||),$$

where **a.b** stands for the dot product of **a** and **b**, **||a||** is the norm of the vector **a** and **arccos** is the inverse cosine function.

Let the vector **a** be the same as defined above and let

```
b = (1:5)'

b =
     1
     2
     3
     4
     5
```

Then

```
angle = acos((a*b)/(norm(a)*norm(b)))

angle =
    1.1303
```

Concept of the cross product can be generalized easily to the set consisting of **n -1** vectors in the n-dimensional Euclidean space $\mathbb{R}^n$. Function **crossprod** provides a generalization of the MATLAB function **cross**.

```
function cp = crossprod(A)

% Cross product cp of a set of vectors that are stored in columns of A.

[n, m] = size(A);
if n ~= m+1
   error('Number of columns of A must be one less than the number of
rows')
end
if rank(A) < min(m,n)
   cp = zeros(n,1);
else
   C = [ones(n,1) A]';
   cp = zeros(n,1);
   for j=1:n
      cp(j) = cofact(C,1,j);
   end
end
```

Let

```
A = [1 -2 3; 4 5 6; 7 8 9; 1 0 1]

A =
     1    -2     3
     4     5     6
     7     8     9
     1     0     1
```

The cross product of column vectors of **A** is

```
cp = crossprod(A)

cp =
    -6
    20
   -14
    24
```

Vector **cp** is orthogonal to the column space of the matrix **A**. One can easily verify this by computing the vector-matrix product

```
cp'*A

ans =
     0     0     0
```

## 3.8 Linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$

Let $\mathbf{L}: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be *a linear transformation*. It is well known that any linear transformation in question is represented by an m-by-n matrix **A**, i.e., **L(x) = Ax** holds true for any $\mathbf{x} \in \mathbb{R}^n$. Matrices of some linear transformations including those of *reflections* and *rotations* are discuss in detail in Tutorial 4, Section 4.3.

With each matrix one can associate four subspaces called the *four fundamental subspaces*. The subspaces in question are called the *column space*, the *nullspace*, the *row space*, and the *left* *nullspace*. First two subspaces are tied closely to the linear transformations on the finite-dimensional spaces.

Throughout the sequel the symbols **$\mathcal{R}$(L)** and **$\mathcal{N}$(L)** will stand for the *range* and the *kernel* of the linear transformation **L**, respectively. Bases of these subspaces can be computed easily. Recall that **$\mathcal{R}$(L) = column space of A** and **$\mathcal{N}$(L) = nullspace of A**. Thus the problem of computing the bases of the range and the kernel of a linear transformation **L** is equivalent to the problem of finding bases of the column space and the nullspace of a matrix that represents transformation **L**.

Function **fourb** uses two MATLAB functions **rref** and **null** to campute bases of four fundamental subspaces associated with a matrix **A**.

```
function [cs, ns, rs, lns] = fourb(A)

% Bases of four fundamental vector spaces associated
% with the matrix A.
% cs-  basis of the column space of A
% ns-  basis of the nullspace of A
% rs-  basis of the row space of A
% lns- basis of the left nullspace of A

[V, pivot] = rref(A);
r = length(pivot);
cs = A(:,pivot);
ns = null(A,'r');
rs = V(1:r,:)';
lns = null(A','r');
```

In this example we will find bases of four fundamental subspaces associated with the random matrix of zeros and ones.
This set up the *seed* of the **randn** function to **0**

```
randn('seed',0)
```

Recall that this function generates normally distributed random numbers. Next a 3-by-5 random matrix is generated using function **randn**

```
A = randn(3,5)

A =
    1.1650    0.3516    0.0591    0.8717    1.2460
    0.6268   -0.6965    1.7971   -1.4462   -0.6390
    0.0751    1.6961    0.2641   -0.7012    0.5774
```

The following trick creates a matrix of zeros and ones from the random matrix **A**

```
A = A >= 0

A =
     1     1     1     1     1
     1     0     1     0     0
     1     1     1     0     1
```

Bases of four fundamental subspaces of matrix **A** are now computed using function **fourb**

```
[cs, ns, rs, lns] = fourb(A)

cs =
     1     1     1
     1     0     0
     1     1     0
ns =
    -1     0
     0    -1
     1     0
     0     0
     0     1
rs =
     1     0     0
     0     1     0
     1     0     0
     0     0     1
     0     1     0
lns =
   Empty matrix: 3-by-0
```

Vectors that form bases of the subspaces under discussion are saved as the column vectors.
The *Fundamental Theorem of Linear Algebra* states that the row space of **A** is orthogonal to the nullspace of **A** and also that the column space of **A** is orthogonal to the left nullspace of **A** (see [6] ). For the bases of the subspaces in this example we have

```
rs'*ns

ans =
     0     0
     0     0
     0     0
```

```
cs'*lns

ans =
   Empty matrix: 3-by-0
```

## 3.9 Real vector spaces

In this section we discuss some computational tools that can be used in studies of real vector spaces. Focus is on linear span, linear independence, transition matrices and the Gram-Schmidt orthogonalization.

### Linear span

Concept of the *linear span* of a set of vectors in a vector space is one of the most important ones in linear algebra. Using MATLAB one can determine easily whether or not given vector is in the span of a set of vectors. Function **span** takes a vector, say **v**, and an unspecified numbers of vectors that form a span. All inputted vectors must be of the same size. On the output a message is displayed to the screen. It says that either **v** is in the span or that **v** is not in the span.

```
function span(v, varargin)

% Test whether or not vector v is in the span of a set
% of vectors.

A = [];
n = length(varargin);
for i=1:n
   u = varargin{i};
   u = u';
   A = [A u(:)];
end
v = v';
v = v(:);
if rank(A) == rank([A v])
   disp(' Given vector is in the span.')
else
   disp(' Given vector is not in the span.')
end
```

The key fact used in this function is a well-known result regarding existence of a solution to the system of linear equations. Recall that the system of linear equations **Ax = b** possesses a solution iff **rank(A) = rank( [A b] )**. MATLAB function **varargin** used here allows a user to enter a variable number of vectors of the span.

To test function **span** we will run this function on matrices. Let

```
v = ones(3);
```

and choose matrices

```
A = pascal(3);
```

and

```
B = rand(3);
```

to determine whether or not **v** belongs to the span of **A** and **B**. Executing function **span** we obtain

```
span(v, A, B)

 Given vector is not in the span.
```

### Linear independence

Suppose that one wants to check whether or not a given set of vectors is *linearly independent*. Utilizing some ideas used in function **span** one can write his/her function that will take an uspecified number of vectors and return a message regarding linear independence/dependence o the given set of vectors. We leave this task to the reader (see Problem 32).

### Transition matrix

Problem of finding the *transition matrix* from one vector space to another vector space is intere in linear algebra. We assume that the ordered bases of these spaces are stored in columns of matrices **T** and **S**, respectively. Function **transmat** implements a well-known method for findin the transition matrix.

```
function V = transmat(T, S)

% Transition matrix V from a vector space having the ordered
% basis T to another vector space having the ordered basis S.
% Bases of the vector spaces are stored in columns of the
% matrices T and S.

[m, n] = size(T);
[p, q] = size(S);
   if (m ~= p) | (n ~= q)
         error('Matrices must be of the same dimension')
   end
V = rref([S T]);
V = V(:,(m + 1):(m + n));
```

Let

```
T = [1 2;3 4]; S = [0 1;1 0];
```

be the ordered bases of two vector spaces. The transition matrix **V** form a vector space having t ordered basis **T** to a vector space whose ordered basis is stored in columns of the matrix **S** is

```
V = transmat(T, S)

V =
     3     4
     1     2
```

We will use the transition matrix **V** to compute a coordinate vector in the basis **S**. Let

$$[\mathbf{x}]_T = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

be the coordinate vector in the basis **T**. Then the coordinate vector $[\mathbf{x}]_S$, is

```
xs = V*[1;1]
```

```
xs =
     7
     3
```

**Gram-Schmidt orthogonalization**

Problem discussed in this subsection is formulated as follows. Given a basis **A = {$u_1$, $u_2$, … , $u_m$}** of a nonzero subspace **W** of $\mathbb{R}^n$. Find an orthonormal basis **V = {$v_1$, $v_2$, … , $v_m$}** for **W**.
Assume that the basis **S** of the subspace **W** is stored in columns of the matrix **A**, i.e.,
**A = [$u_1$; $u_2$; … ; $u_m$]**, where each **$u_k$** is a column vector. Function **gs(A)** computes an orthonormal basis **V** for **W** using a classical method of Gram and Schmidt.

```
function V = gs(A)

% Gram-Schmidt orthogonalization of vectors stored in
% columns of the matrix A. Orthonormalized vectors are
% stored in columns of the matrix V.

[m,n] = size(A);
for k=1:n
   V(:,k) = A(:,k);
   for j=1:k-1
      R(j,k) = V(:,j)'*A(:,k);
      V(:,k) = V(:,k) - R(j,k)*V(:,j);
   end
   R(k,k) = norm(V(:,k));
   V(:,k) = V(:,k)/R(k,k);
end
```

Let **W** be a subspace of $\mathbb{R}^3$ and let the columns of the matrix **A**, where

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}$$

form a basis for **W**. An orthonormal basis **V** for **W** is computed using function **gs**

```
V = gs([1 1;2 1;3 1])

V =
    0.2673    0.8729
    0.5345    0.2182
    0.8018   -0.4364
```

To verify that the columns of **V** form an orthonormal set it sufficies to check that $\mathbf{V}^T\mathbf{V} = \mathbf{I}$. We have

```
V'*V

ans =
    1.0000    0.0000
    0.0000    1.0000
```

We will now use matrix **V** to compute the coordinate vector **$[v]_V$**, where

```
v = [1 0 1];
```

We have

```
v*V

ans =
    1.0690    0.4364
```

## 3.10 The matrix eigenvalue problem

MATLAB function **eig** is designed for computing the eigenvalues and the eigenvectors of the matrix **A**. Its syntax is shown below

**[V, D] = eig(A)**

The eigenvalues of **A** are stored as the diagonal entries of the diagonal matrix **D** and the associated eigenvectors are stored in columns of the matrix **V**.

Let

```
A = pascal(3);
```

Then

```
[V, D] = eig(A)

V =
    0.5438   -0.8165    0.1938
   -0.7812   -0.4082    0.4722
    0.3065    0.4082    0.8599
D =
    0.1270         0         0
         0    1.0000         0
         0         0    7.8730
```

Clearly, matrix **A** is *diagonalizable*. The *eigenvalue-eigenvector decomposition* $\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$ of is calculated as follows

```
V*D/V
```

```
ans =
    1.0000    1.0000    1.0000
    1.0000    2.0000    3.0000
    1.0000    3.0000    6.0000
```

Note the use of the *right division operator* **/** instead of using the inverse matrix function **inv**. This is motivated by the fact that computation of the inverse matrix takes longer than the execution of the right division operation.

The *characteristic polynomial* of a matrix is obtained by invoking the function **poly**.
Let

```
A = magic(3);
```

be the *magic square*. In this example the vector **chpol** holds the coefficients of the *characteristic polynomial* of the matrix **A**. Recall that a polynomial is represented in MATLAB by its coefficients that are ordered by descending powers

```
chpol = poly(A)

chpol =
    1.0000  -15.0000  -24.0000  360.0000
```

The eigenvalues of **A** can be computed using function **roots**

```
eigenvals = roots(chpol)

eigenvals =
   15.0000
    4.8990
   -4.8990
```

This method, however, is not recommended for numerical computing the eigenvalues of a matrix. There are several reasons for which this approach is not used in numerical linear algebra. An interested reader is referred to Tutorial 4.

The *Caley-Hamilton Theorem* states that each matrix satisfies its characteristic equation, i.e., **chpol(A) = 0**, where the last zero stands for the matrix of zeros of the appropriate dimension. We use function **lincomb** to verify this result

```
Q = lincomb(num2cell(chpol), {A^3, A^2, A, eye(size(A))})

Q =
  1.0e-012 *
   -0.5684   -0.5542   -0.4832
   -0.5258   -0.6253   -0.4547
   -0.5116   -0.4547   -0.6821
```

## 3.11 Applications of Linear Algebra

List of applications of methods of linear algebra is long and impressive. Areas that relay heavil on the methods of linear algebra include the data fitting, mathematical statistics, linear programming, computer graphics, cryptography, and economics, to mention the most importan ones. Applications discussed in this section include the data fitting, coding messages, and computer graphics.

### Data fitting

In many problems that arise in science and engineering one wants to fit a discrete set of points i the plane by a smooth curve or function. A typical choice of a smoothing function is a polynom of a certain degree. If the smoothing criterion requires minimization of the 2-norm, then one ha to solve the *least-squares approximation problem*. Function **fit** takes three arguments, the degre of the approximating polynomial, and two vectors holding the x- and the y- coordinates of poin to be approximated. On the output, the coefficients of the least-squares polynomials are returne Also, its graph and the plot of the data points are generated.

```
function c = fit(n, t, y)

% The least-squares approximating polynomial of degree n (n>=0).
% Coordinates of points to be fitted are stored in the column vectors
% t and y. Coefficients of the approximating polynomial are stored in
% the vector c. Graphs of the data points and the least-squares
% approximating polynomial are also generated.

if ( n >= length(t))
   error('Degree is too big')
end
v = fliplr(vander(t));
v = v(:,1:(n+1));
c = v\y;
c = fliplr(c');
x = linspace(min(t),max(t));
w = polyval(c, x);
plot(t,y,'ro',x,w);
title(sprintf('The least-squares polynomial of degree n = %2.0f',n))
legend('data points','fitting polynomial')
```

To demonstrate functionality of this code we generate first a set of points in the plane. Our goal to fit ten evenly spaced points with the y-ordinates being the values of the function **y = sin(2t)** a these points

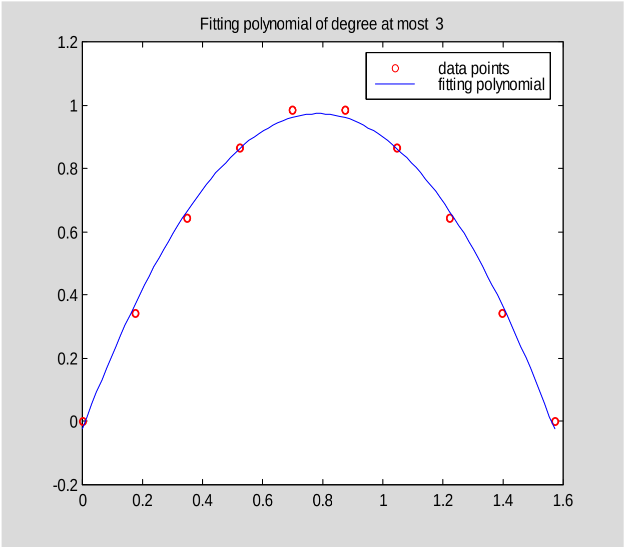
```
t = linspace(0, pi/2, 10); t = t';

y = sin(2*t);
```

We will fit the data by a polynomial of degree at most three

```
c = fit(3, t, y)

c =
   -0.0000   -1.6156    2.5377   -0.0234
```




## Coded messages

Some elementary tools of linear algebra can be used to code and decode messages. A typical message can be represented as a string. The following **'coded message'** is an example of the string in MATLAB. Strings in turn can be converted to a sequence of positive integers using MATLAB's function **double**. To code a transformed message multiplication by a nonsingular matrix is used. Process of decoding messages can be viewed as the inverse process to the one described earlier. This time multiplication by the inverse of the coding matrix is applied and next MATLAB's function **char** is applied to the resulting sequence to recover the original message. Functions **code** and **decode** implement these steps.

```
function B = code(s, A)

% String s is coded using a nonsingular matrix A.
% A coded message is stored in the vector B.

p = length(s);
[n,n] = size(A);
b = double(s);
r = rem(p,n);
if r ~= 0
   b = [b zeros(1,n-r)]';
end
b = reshape(b,n,length(b)/n);
B = A*b;
B = B(:)';
```

```
function s = dcode(B, A)

% Coded message, stored in the vector B, is
% decoded with the aid of the nonsingular matrix A
% and is stored in the string s.

[n,n]= size(A);
p = length(B);
B = reshape(B,n,p/n);
d = A\B;
s = char(d(:)');
```

A message to be coded is

```
s = 'Linear algebra is fun';
```

As a coding matrix we use the Pascal matrix

```
A = pascal(4);
```

This codes the message **s**

```
B = code(s,A)
```

```
B =
  Columns 1 through 6
         392        1020        2061        3616         340
809
  Columns 7 through 12
        1601        2813         410        1009        2003
3490
  Columns 13 through 18
         348         824        1647        2922         366
953
  Columns 19 through 24
        1993        3603         110         110         110
110
```

To decode this message we have to work with the same coding matrix **A**

```
dcode(B,A)
```

```
ans =
Linear algebra is fun
```

## Computer graphics

Linear algebra provides many tools that are of interest for computer programmers especially fo those who deal with the computer graphics. Once the graphical object is created one has to transform it to another object. Certain plane and/or space transformations are linear. Therefore they can be realized as the matrix-vector multiplication. For instance, the reflections, translatio rotations all belong to this class of transformations. A computer code provided below deals with the plane rotations in the counterclockwise direction. Function **rot2d** takes a planar object represented by two vectors **x** and **y** and returns its image. The angle of rotation is supplied in the degree measure.

```
function [xt, yt] = rot2d(t, x, y)

% Rotation of a two-dimensional object that is represented by two
% vectors x and y. The angle of rotation t is in the degree measure.
% Transformed vectors x and y are saved in xt and yt, respectively.

t1 = t*pi/180;
r = [cos(t1) -sin(t1);sin(t1) cos(t1)];
x = [x x(1)];
y = [y y(1)];
hold on
grid on
axis equal
fill(x, y,'b')
z = r*[x;y];
xt = z(1,:);
yt = z(2,:);
fill(xt, yt,'r');
title(sprintf('Plane rotation through the angle of %3.2f degrees',t))
hold off
```

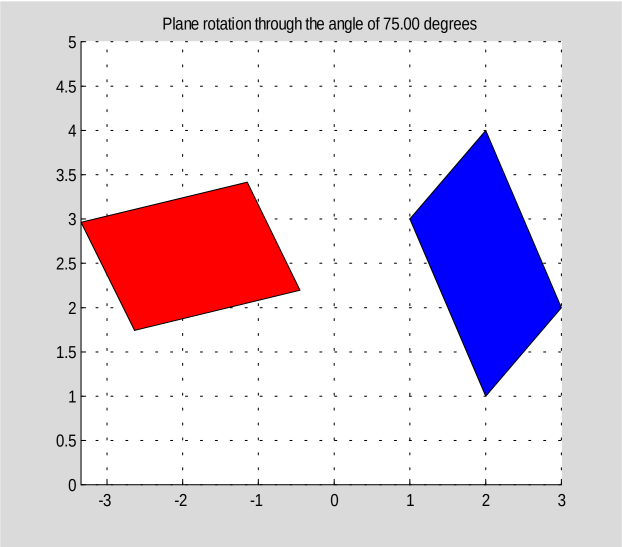
Vectors **x** and **y**

```
x = [1 2 3 2]; y = [3 1 2 4];
```

are the vertices of the parallelogram. We will test function **rot2d** on these vectors using as the angle of rotation **t = 75**.

```
[xt, yt] = rot2d(75, x, y)
```

```
xt =
   -2.6390   -0.4483   -1.1554   -3.3461   -2.6390
yt =
    1.7424    2.1907    3.4154    2.9671    1.7424
```




The right object is the original parallelogram while the left one is its image.

## References


[1] B.D. Hahn, Essential MATLAB for Scientists and Engineers, John Wiley & Sons, New York, NY, 1997.

[2] D.R. Hill and D.E. Zitarelli, Linear Algebra Labs with MATLAB, Second edition, Prentice Hall, Upper Saddle River, NJ, 1996.

[3] B. Kolman, Introductory Linear Algebra with Applications, Sixth edition, Prentice Hall, Upper Saddle River, NJ, 1997.

[4] R.E. Larson and B.H. Edwards, Elementary Linear Algebra, Third edition, D.C. Heath and Company, Lexington, MA, 1996.

[5] S.J. Leon, Linear Algebra with Applications, Fifth edition, Prentice Hall, Upper Saddle River, NJ, 1998.

[6] G. Strang, Linear Algebra and Its Applications, Second edition, Academic Press, Orlando, FL, 1980.


## Problems

In Problems 1 – 12 you cannot use loops **for** and/or **while**.
Problems 40 - 42 involve symbolic computations. In order to do these problems you have to us the **Symbolic Math Toolbox**.

1. Create a ten-dimensional row vector whose all components are equal **2**. You cannot enter number **2** more than once.

2. Given a row vector **a = [1 2 3 4 5]**. Create a column vector **b** that has the same components the vector **a** but they must bestored in the reversed order.

3. MATLAB built-in function **sort(a)** sorts components of the vector **a** in the ascending order Use function **sort** to sort components of the vector **a** in the descending order.

4. To find the largest (smallest) entry of a vector you can use function **max** (**min**). Suppose th these functions are not available. How would you calculate

   (a) the largest entry of a vector ?
   (b) the smallest entry of a vector?

5. Suppose that one wants to create a vector **a** of ones and zeros whose length is equal to **2n** ( **n = 1, 2, …** ). For instance, when **n = 3**, then **a = [1 0 1 0 1 0]**. Given value of **n** create a vector **a** with the desired property.

6. Let **a** be a vector of integers.

   (a) Create a vector **b** whose all components are the even entries of the vector **a**.
   (b) Repeat part (a) where now **b** consists of all odd entries of the vector **a**.

   Hint: Function **logical** is often used to logical tests. Another useful function you may consider to use is **rem(x, y)** - the remainder after division of **x** by **y**.

7. Given two nonempty row vectors **a** and **b** and two vectors **ind1**and **ind2** with **length(a) = length(ind1)** and **length(b) = length(ind2)**. Components of **ind1** and **ind2** are positive integers. Create a vector **c** whose components are those of vectors **a** and **b**. Their indices ar determined by vectors **ind1** and **ind2**, respectively.

8. Using function **rand**, generate a vector of random integers that are uniformly distributed in the interval **(2, 10)**. In order to insure that the resulting vector is not empty begin with a vector that has a sufficient number of components.
   Hint: Function **fix** might be helpful. Type **help fix** in the **Command Window** to learn mor about this function.

9. Let **A** be a square matrix. Create a matrix **B** whose entries are the same as those of **A** excep the entries along the main diagonal. The main diagonal of the matrix **B** should consist entie of ones.

10. Let **A** be a square matrix. Create a tridiagonal matrix **T** whose subdiagonal, main diagonal, and the superdiagonal are taken from the matrix **A**.
Hint: You may wish to use MATLAB functions **triu** and **tril**. These functions take a second optional argument. To learn more about these functions use MATLAB's help.

11. In this exercise you are to test a square matrix **A** for symmetry. Write MATLAB function **s = issymm(A)** that takes a matrix **A** and returns a number **s**. If **A** is symmetric, then **s = 1**, otherwise **s = 0**.

12. Let **A** be an m-by-n and let **B** be an n-by-p matrices. Computing the product **C = AB** requires **mnp** multiplications. If either **A** or **B** has a special structure, then the number of multiplications can be reduced drastically. Let **A** be a full matrix of dimension m-by-n and let **B** be an upper triangular matrix of dimension n-by-n whose all nonzero entries are equal to one. The product **AB** can be calculated without using a single multiplicationa. Write an algorithm for computing the matrix product **C = A*B** that does not require multiplications. Test your code with the following matrices **A = pascal(3)** and **B = triu(ones(3))**.

13. Given square invertible matrices **A** and **B** and the column vector **b**. Assume that the matrices **A** and **B** and the vector **b** have the same number of rows. Suppose that one wants to solve a linear system of equations **ABx = b**. Without computing the matrix-matrix product **A*B**, find a solution **x** to to this system using the backslash operator ****.

14. Find all solutions to the linear system **Ax = b**, where the matrix **A** consists of rows one through three of the 5-by-5 magic square

```
A = magic(5);

A = A(1:3,: )

A =
    17    24     1     8    15
    23     5     7    14    16
     4     6    13    20    22
```

and **b = ones(3; 1)**.

15. Determine whether or not the system of linear equations **Ax = b**, where

```
A = ones(3, 2);  b = [1; 2; 3];
```

possesses an exact solution **x**.

16. The purpose of this exercise is to demonstrate that for some matrices the computed solution to **Ax = b** can be poor. Define

```
A = hilb(50);   b = rand(50,1);
```

Find the 2-norm of the residual **r = A*x – b**. How would you explain a fact that the computed norm is essentially bigger than zero?

17. In this exercise you are to compare computational complexity of two methods for finding a solution to the linear system **Ax = b** where **A** is a square matrix. First method utilizes the backslash operator **** while the second method requires a use of the function **rref**. Use MATLAB function **flops** to compare both methods for various linear systems of your choic
Which of these methods require, in general, a smaller number of flops?

18. Repeat an experiment described in Problem 17 using as a measure of efficiency a time need to compute the solution vector. MATLAB has a pair of functions **tic** and **toc** that can be use in this experiment. This illustrates use of the above mentioned functions **tic; x = A\b; toc**. Using linear systems of your choice compare both methods for speed. Which method is a faster one? Experiment with linear systems having at least ten equations

19. Let **A** be a real matrix. Use MATLAB function **rref** to extract all

(a) columns of **A** that are linearly independent
(b) rows of **A** that are linearly independent

20. In this exercise you are to use MATLAB function **rref** to compute the rank of the followin matrices:

**(a) A = magic(3)**
**(b) A = magic(4)**
**(c) A = magic(5)**
**(d) A = magic(6)**

Based on the results of your computations what hypotheses would you formulate about the **rank(magic(n))**, when **n** is odd, when **n** is even?

21. Use MATLAB to demonstrate that **det(A + B) ≠ det(A) + det(B)** for matrices of your choi

22. Let **A = hilb(5)**. Hilbert matrix is often used to test computer algorithms for reliability. In t exercise you will use MATLAB function **num2str** that converts numbers to strings, to see that contrary to the well-known theorem of Linear Algebra the computed determinant **det(A*A')** is not necessarily the same as **det(A)*det(A')**. You can notice a difference in computed quantities by executing the following commands: **num2str(det(A*A'), 16)** and **num2str(det(A)*det(A'), 16)**.

23. The inverse matrix of a symmetric nonsingular matrix is a symmetric matrix. Check this property using function **inv** and a symmetric nonsingular matrix of your choice.

24. The following matrix

```
A = ones(5) + eye(5)

A =
     2     1     1     1     1
     1     2     1     1     1
     1     1     2     1     1
     1     1     1     2     1
     1     1     1     1     2
```

is a special case of the *Pei matrix*. Normalize columns of the matrix **A** so that all columns of the resulting matrix, say **B**, have the Euclidean norm (2-norm) equal to one.

25. Find the angles between consecutive columns of the matrix **B** of Problem 24.

26. Find the cross product vector **cp** that is perpendicular to columns one through four of the Pei matrix of Problem 24.

27. Let **L** be a linear transformation from $\mathbb{R}^5$ to $\mathbb{R}^5$ that is represented by the Pei matrix of Problem 24. Use MATLAB to determine the range and the kernel of this transformation.

28. Let $\mathbb{P}_n$ denote a space of algebraic polynomials of degree at most **n**. Transformation **L** from $\mathbb{P}_n$ to $\mathbb{R}^3$ is defined as follows

$$\mathbf{L(p)} = \begin{bmatrix} \int_0^1 \mathbf{p(t)dt} \\ \mathbf{p(0)} \\ \mathbf{0} \end{bmatrix}$$

(a) Show that **L** is a linear transformation.
(b) Find a matrix that represents transformation **L** with respect to the ordered basis $\{t^n, t^{n-1}, \dots 1\}$.
(c) Use MATLAB to compute bases of the range and the kernel of **L**. Perform your experiment for the following values of **n = 2, 3, 4**.

29. Transformation **L** from $\mathbb{P}_n$ to $\mathbb{P}_{n-1}$ is defined as follows **L(p) = p'(t)**. Symbol $\mathbb{P}_n$, is introduced in Problem 28. Answer questions (a) through (c) of Problem 28 for the transformation **L** of this problem.

30. Given vectors **a = [1; 2; 3]** and **b = [-3; 0; 2]**. Determine whether or not vector **c = [4; 1;1]** is in the span of vectors **a** and **b**.

31. Determine whether or not the Toeplitz matrix

```
A = toeplitz( [1 0 1 1 1] )

A =
     1     0     1     1     1
     0     1     0     1     1
     1     0     1     0     1
     1     1     0     1     0
     1     1     1     0     1
```

is in the span of matrices **B = ones(5)** and **C = magic(5)**.

32. Write MATLAB function **linind(varargin)** that takes an arbitrary number of vectors (matrices) of the same dimension and determines whether or not the inputted vectors (matrices) are linearly independent. You may wish to reuse some lines of code that are contained in the function **span** presented in Section 3.9 of this tutorial.

33. Use function **linind** of Problem 32 to show that the columns of the matrix **A** of Problem 31 are linearly independent.

34. Let $\mathbf{[a]_A}$ **= ones(5,1)** be the coordinate vector with respect to the basis **A** – columns of the matrix **A** of Problem 31. Find the coordinate vector $\mathbf{[a]_P}$ , where **P** is the basis of the vector space spanned by the columns of the matrix **pascal(5)**.

35. Let **A** be a real symmetric matrix. Use the well-known fact from linear algebra to determin the interval containing all the eigenvalues of **A**. Write MATLAB function **[a, b] = interval(A)** that takes a symmetric matrix **A** and returns the endpoints **a** and **b** of t interval that contains all the eigenvalues of **A**.

36. Without solving the matrix eigenvalue problem find the sum and the product of all eigenvalues of the following matrices:

**(a)** **P = pascal(30)**
**(b)** **M= magic(40)**
**(c)** **H = hilb(50)**
**(d)** **H = hadamard(64)**

37. Find a matrix **B** that is similar to **A = magic(3)**.

38. In this exercise you are to compute a power of the diagonalizable matrix **A**. Let **A = pascal(5)**. Use the eigenvalue decomposition of **A** to calculate the ninth power of **A**. You cannot apply the power operator **^** to the matrix **A**.

39. Let **A** be a square matrix. A matrix **B** is said to be the *square root* of **A** if **B^2 = A**. In MATLAB the square root of a matrix can be found using the power operator **^**. In this exercise you are to use the eigenvalue-eigenvector decomposition of a matrix find the squa root of **A = [3 3;-2 -2]**.

**40.** Declare a variable **k** to be a symbolic variable typing **syms k** in the **Command Window**. Find a value of **k** for which the following symbolic matrix **A = sym( [1 k^2 2; 1 k -1; 2 –1 0] )** is not invertible.

41. Let the matrix **A** be the same as in Problem 40.

(a) Without solving the matrix eigenvalue problem, determine a value of **k** for which all t eigenvalues of **A** are real.
(b) Let **v** be a number you found in part (a). Convert the symbolic matrix **A** to a numeric matrix **B** using the substitution command **subs**, i.e., **B = subs(A, k, v)**.
(c) Determine whether or not the matrix **B** is diagonalizable. If so, find a diagonal matrix that is similar to **B**.

(d) If matrix **B** is diagonalizable use the results of part (c) to compute all the eigenvectors of the matrix **B**. Do not use MATLAB's function **eig**.

42. Given a symbolic matrix **A = sym( [1 0 k; 2 2 0; 3 3 3])**.

(a) Find a nonzero value of **k** for which all the eigenvalues of **A** are real.
(b) For what value of **k** two eigenvalues of **A** are complex and the remaining one is real?