Randomized Algorithms
ITU Copenhagen, Fall 2004
Exercises and hand-in

G. S. Frandsen, A. Pagh, and R. Pagh

September 22, 2004

Exercises on September 28, 13-15

Motwani-Raghavan exercises 2.3, 2.7, 4.8, 4.12, 4.13.

Hand-in (deadline: Lecture on October 5)

Algorithm FIND in problem 1 is recursive. If the recursion depth of an execution of FIND is large, then the recursion stack may take up a lot of space. However, this is unlikely to happen.

> In this problem you are required to prove that the probability that the recursion depth exceeds $c \log n$ in an execution of FIND is at most $e^{-d \log n}$ for some explicit constants $c, d \geq 1$. You must use the Chernoff bound technique (see Motwani and Raghavan 4.1).

Below is defined some notation and set up some subproblems that may be of help.

Notation: Let n_i denote the length of the input list to the ith recursive call and define the indicator variables

$$X_i = \begin{cases} 1, & \text{if the recursion reaches depth } i+1 \text{ and } n_{i+1} > \frac{3}{4}n_i \\ 0, & \text{otherwise} \end{cases}$$

Subproblem: Explain a connection between $\sum_i X_i$ and the recursion depth. In particular, it may be helpful to prove that if the recursion depth is at least $c \log n$ then

$$\sum_{i=1}^{c \log n} X_i \geq c \log n - \log_{4/3} n$$

Subproblem: Are $\{X_i\}$ independent random variables?

Notation: Define the indicator variables

$$Z_i = \begin{cases} 1, & \text{if the recursion reaches depth } i+1 \text{ and the random choice} \\ & \text{in the } i\text{th recursion step is among that half of the possible} \\ & \text{choices that leads to largest values of } n_{i+1}. \\ & \text{Or if the recursion does not reach depth } i+1 \text{, but the toss} \\ & \text{of a fair coin yelds ``heads''.} \\ 0, & \text{otherwise} \end{cases}$$

Subproblem: What is $E[Z_i]$?

Subproblem: Are $\{Z_i\}$ independent random variables? (give an argument for your answer)

Subproblem: Describe a simple relation between X_i and Z_i.

Subproblem: Argue that the probability that the recursion depth exceeds $c \log n$ is bounded by

$$\text{Prob}(\sum_{i=1}^{c \log n} Z_i > (2 - \frac{2}{c \log \frac{4}{3}})\text{E}[\sum_{i=1}^{c \log n} Z_i])$$

Subproblem: Explain how a Chernoff bound can give you the wanted result. Be careful to specify which version of the Chernoff bound that you use, and be careful to argue that all preconditions for applying the Chernoff bound are satisfied. Argue that it suffices to substitute an explicit value for c and then bound the probability in the previous subproblem. For what values of c, can a Chernoff bound give you the wanted result?