Cartesian Closed Dialectica Categories

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Abstract

When Gödel developed his functional interpretation, also known as the Dialectica interpretation, his aim was to prove (relative) consistency of first order arithmetic by reducing it to a quantifier-free theory with finite types. Like other functional interpretations (e.g. Kleene’s realizability interpretation and Kreisel’s modified realizability) Gödel’s Dialectica interpretation gives rise to category theoretic constructions that serve both as new models for logic and semantics and as tools for analysing and understanding various aspects of the Dialectica interpretation itself.

Gödel’s Dialectica interpretation gives rise to the Dialectica categories (described by V. de Paiva in [dP89] and J.M.E. Hyland in [Hyl02]). These categories are symmetric monoidal closed and have finite products and weak coproducts, but they are not Cartesian closed in general. We give an analysis of how to obtain weakly Cartesian closed and Cartesian closed Dialectica categories, and we also reflect on what the analysis might tell us about the Dialectica interpretation.

1 Introduction

In this paper we analyse how to obtain Cartesian Closed Dialectica categories. The inspiration for exponent construction that we will give comes from a structure closely related to the Dialectica categories, namely the Dialectica tripos [BBLBCB07] (which is actually an indexed, preordered reflection of a Dialectica category). In order to do this analysis and also to find out whether the construction of the exponential in the tripos can be carried over to the Dialectica categories to give some sort of exponential in these, we first generalise the original Dialectica categories to include fibrations for type theory. In [dP89], Valeria de Paiva explores the Dialectica categories using the subobject fibration, Martin Hyland generalises the definition of a Dialectica category in [Hyl02] to include other preordered fibrations. In this paper we also include fibrations for type theory, that is, fibrations where the fibres are general categories instead of preorders. We will focus on a case study, namely the codomain fibration. The main reason for considering Dialectica categories over general fibrations is that when we start out with more structure, we are forced to be less flexible and the nature of the structures we are studying will reveal themselves. As a spin-off we get a whole new class of Dialectica categories. The analysis shows that both the original Dialectica categories and the Dialectica categories for type theory have a weak exponential, so together with the Cauchy completion we get Cartesian closed Dialectica categories.

Outline of the paper: We start by recalling the definition and closure properties of V. de Paiva’s and J.M.E. Hyland’s Dialectica categories. We then indicate three different approaches to obtain classes of Cartesian closed Dialectica categories one of which will be studied in this
paper. Next we define a generalised version of Dialectica categories and show that they have products. In Section 4 we study monads leading to comonads on Dialectica categories, and we describe the Kleisli category in the general setting. If a comonad is Girardian, we automatically get a Cartesian closed Kleisli category for the comonad. In the most technical part of the paper we show that for a particular non-Girardian comonad, $L^+$ applied to a Dialectica category $\text{Dial}(\text{cod}(C))$, gives a Kleisli category, $\text{Dial}^+$ with weak exponentials. This result also holds for the original Dialectica categories, $\text{Dial}(\text{Sub}(C))$ (Dialectica categories over the subobject fibration). This implies that the Cauchy completion of $\text{Dial}^+$ is Cartesian closed and also that the preordered reflection of $\text{Dial}^+$ is a Heyting algebra. Finally, we spell out the details of an example that might be of particular interest since it corresponds to an extensional version of Dialectica.

2 The Dialectica Categories

In this section we recall the definition of Dialectica categories and their closure properties as given in [dP89] and [Hyl02]. The following is quoted from [Hyl02]: Suppose that we have a category $T$ which we can think of as interpreting some type theory; and suppose that over the category $T$ we have a preordered fibration $p : E \to T$, which we can regard as providing for each $I \in T$ a preordered collection of (possibly non-standard) predicates $\mathcal{E}(I) = (\mathcal{E}(I), \vdash)$. Starting with this data we construct a new category $\text{Dial}(p)$ which we regard as a category of propositions and proofs.

We do this as follows.

- The objects $A$ of $\text{Dial}(p)$ are $U, X \in T$ together with $\alpha \in \mathcal{E}(U \times X)$. We write this as $A = U \xrightarrow{\alpha} X$. Our understanding of the predicate $\alpha$ is not symmetric as regards $U$ and $X$: we read $U \xrightarrow{\alpha} X$ as $\exists u. \forall x. \alpha(u, x)$, in accord with the form of propositions in the image of the Dialectica interpretation.

- Maps of $\text{Dial}(p)$ from $A = U \xrightarrow{\alpha} X$ to $B = V \xrightarrow{\beta} Y$ are diagrams of the form

$$
\begin{array}{ccc}
U & \xrightarrow{\alpha} & X \\
\downarrow{f} & & \downarrow{F} \\
V & \xleftarrow{\beta} & Y
\end{array}
$$

with $\alpha(u, F(u, y)) \vdash \beta(f(u), y)$ in $\mathcal{E}(U \times Y)$.

Thus maps $A \to B$ of $\text{Dial}(p)$ consists of maps $f : U \to V$ and $F : U \times Y \to X$ in $T$ such that $\alpha(u, F(u, y)) \vdash \beta(f(u), y)$ holds in $\mathcal{E}(U \times Y)$.

**Proposition 2.1.** $\text{Dial}(p)$ forms a category

The original Dialectica categories described in [dP89] were defined only with $p$ being the subobject fibration. And here comes the closure properties:
Proposition 2.2. If \( p : \mathcal{E} \to \mathcal{T} \) is a product fibration, i.e., \( \mathcal{T} \) has finite products and the fibres \( \mathcal{E}(I) \) have finite products preserved by reindexing then \( \text{Dial}(p) \) is a symmetric monoidal category.

Proposition 2.3. If \( \mathcal{T} \) is ccc and \( p \) is fibered Cartesian closed then \( \text{Dial}(p) \) is symmetric monoidal closed.

Proposition 2.4. If \( \mathcal{T} \) has finite, distributive coproducts and \( \mathcal{E}(0) \cong 1 \) and the injections \( X \to X + Y \) and \( Y \to X + Y \) induce an equivalence \( \mathcal{E}(X + Y) \cong \mathcal{E}(X) \times \mathcal{E}(Y) \) then \( \text{Dial}(p) \) has finite products.

2.1 Cartesian Closed Dialectica Categories

We now describe three different approaches to obtain Cartesian closed Dialectica categories. We have seen that the natural structure of the category \( \text{Dial}(p) \) is smcc with finite products. One way to obtain Cartesian closure is by adding structure that will make \( \otimes \) a product, that is, making sure we get projections and diagonals for \( \otimes \). This approach has been studied briefly in [Hyl02]. Another way of obtaining Cartesian closed Dialectica categories is by altering the definition slightly to get variations like the Diller-Nahm Dialectica category (see [dP89]). There are actually several variants constructed in the same manner as the Diller-Nahm category, that is, by a Girardian comonad on the Dialectica category. First steps in this direction has been made in [Bie08], and also implicitly in [Oli08]. The third approach that one might think of is to add enough structure to define an exponent (without making \( \otimes = \land \)). The rest of this paper is devoted to analysing under what circumstances we can get a Cartesian closed Dialectica category using the third approach.

3 Dialectica Categories for Cloven Fibrations

In this section we define a more general version of the Dialectica categories and show that they have finite products.

Given a cloven fibration \( p : \mathcal{E} \to \mathcal{T} \), if \( \mathcal{T} \) has binary products, we can construct the Dialectica category for types over \( p \), written \( \text{Dial}(p) \) as follows:

- Objects are triples \( (U, X, \alpha) \), where \( U, X \in \mathcal{T} \) and \( \alpha \in \mathcal{E} \) is an object in the fibre over \( U \times X \).
- A map from \( (U, X, \alpha) \) to \( (V, Y, \beta) \) is a triple \( (f, F, \varphi) \), where \( f : U \to V \), \( F : U \times Y \to X \) and \( \varphi(u, y) : \alpha(u, F(u, y)) \to \beta(f(u), y) \) in the fibre over \( U \times Y \).

We can think of \( \mathcal{T} \) as our types and \( \alpha \in \mathcal{E}(U \times X) \) as a dependent type over \( U \times X \). Maps are written

\[
\begin{array}{ccc}
U & \xleftarrow{\alpha} & X \\
\downarrow{f} & & \downarrow{F} \\
V & \xleftarrow{\beta} & Y
\end{array}
\quad \phi(u, y) : \alpha(u, F(u, y)) \to \beta(f(u), y)
\]

This forms a category, with
• identity arrows: \((\text{id}_U, \pi_2, \text{id}_{\alpha(u,x)})\).

• Composition works as follows: Given \((f, F, \phi(u, y)) : (U, X, \alpha) \to (V, Y, \beta)\) and \((g, G, \psi(v, w)) : (V, Y, \beta) \to (Z, W, \gamma)\), the composite is

\[(gf, F(u, G(f(u), w)), \psi(f(u), w) \circ \phi(u, G(f(u), w))),\]

where \(F(u, G(f(u), w))\) is the arrow \(U \times W \xrightarrow{(U, f \times W)} U \times (V \times W) \xrightarrow{U \times G} U \times Y \xrightarrow{F} X\), and \(\phi(u, G(f(u), w))\) is reindexing of \(\phi(u, y)\) along the arrow \((U \times G) \circ (U, f \times W)\).

Since reindexing in a fibration is up to isomorphism, we need the fibration to be cloven in order for composition to be associative. Associativity is then a consequence of the coherence conditions for a cloven fibration. Note that this only regards the third component of the arrows in Dial\((p)\).

### 3.1 Products in Dial\((p)\)

**Definition 3.1.** A category \(C\) has finite, distributive coproducts if it has finite coproducts and products, and the product functor preserves coproducts, that is, for objects \(A, X, Y\) there is a natural isomorphism \(\delta : A \times (X + Y) \cong A \times X + A \times Y\) such that the following diagram commutes:

\[
\begin{array}{ccc}
A \times X & \xrightarrow{A \times \iota_X} & A \times (X + Y) \\
\downarrow{\iota_{A \times X}} & & \downarrow{\iota_{A \times Y}} \\
A \times X + A \times Y & \xrightarrow{\delta} & A \times Y
\end{array}
\]

**Fact:** In a distributive category, 0 is a strict initial object.

Of course, any Cartesian closed category with finite coproducts is distributive.

Let \(q : \mathcal{E} \to T\) be a preordered fibration and let \(\mathcal{E}(X)\) denote the fibre over \(X\). From [Hyl02] we know that, if \(T\) has finite, distributive coproducts, and it also holds that \(\mathcal{E}(0) \cong 1\) and that the injections \(X \to X + Y\) and \(Y \to X + Y\) induce an equivalence \(\mathcal{E}(X + Y) \cong \mathcal{E}(X) \times \mathcal{E}(Y)\), then Dial\((q)\) has finite products. We now show that this also holds in the general case, where the fibres \(\mathcal{E}(X)\) are not preorders, but categories.

**Proposition 3.2.** Let \(p : \mathcal{E} \to T\) be a cloven fibration.

1. Suppose \(T\) has finite, distributive coproducts and products, and that the injections \(X \to X + Y\) and \(Y \to X + Y\) induce an equivalence \(\mu : \mathcal{E}(X) \times \mathcal{E}(Y) \cong \mathcal{E}(X + Y)\), natural in \(X, Y\), then Dial\((p)\) has binary products.

2. Suppose that \(\mathcal{E}(0) \cong 1\), then Dial\((p)\) has a terminal object.

**Proof:** This is a straightforward generalization of the proof found in [Hyl02]. First note that since the equivalence \(\mu\) is induced by the injections, we have \(\mu^{-1} = (\iota_X^*, \iota_Y^*)\) and so

\[
\iota_X^* \mu(\phi, \psi) \cong \phi
\]

for \(\phi \in \mathcal{E}(X)\) and \(\psi \in \mathcal{E}(Y)\). The product \(A \times B\) of \(A = (U \xrightarrow{\alpha} X)\) and \(B = (V \xleftarrow{\beta} Y)\) is

\[A \times B = (U \times V \xleftarrow{\alpha \& \beta} X + Y)\]
where $\alpha, \beta \in \mathcal{E}(U \times V \times (X + Y))$ are given by $\delta^* \mu(\alpha(\pi_U(u, v), x), \beta(\pi_V(u, v), y))$. The projections are $(\pi_U, \pi_X, \alpha(\pi_U(u, v), x))$ and $(\pi_V, \pi_Y, \beta(\pi_V(u, v), y))$.

Let $C$ be the object $(\gamma : Z \rightarrow W)$. Given morphisms

$$(f, F, \phi(z, x)) : C \rightarrow A$$

and

$$(g, G, \psi(z, y)) : C \rightarrow B,$$

the universal map from $C$ to $A \times B$ is

$$(\langle f, g \rangle, [F, G], \delta^* \mu(\phi(z, x), \psi(z, y))),$$

where $\delta^* \mu(\phi(z, x), \psi(z, y))) \in \mathcal{E}(Z \times (X + Y))$ is reindexing of $\mu(\phi(z, x), \psi(z, y)) \in \mathcal{E}(Z \times X + Z \times Y)$ along the isomorphism $\delta : Z \times (X + Y) \rightarrow Z \times X + Z \times Y$.

The composite of

$$(\langle f, g \rangle, [F, G], \delta^* \mu(\phi(z, x), \psi(z, y)))$$

with the projection

$$(\pi_U, \iota_X \circ \pi_X, \text{id}_{\alpha(\pi_U(u, v), x)})$$

is the arrow

$$(\pi_u \circ \langle f, g \rangle, [F, G](z, \iota_X \pi_X((\langle f, g \rangle z, x)), \text{id}_{\alpha(\pi_U(u, v), x)}((\langle f, g \rangle z, x) \circ (z, \iota_X \pi_X((\langle f, g \rangle z, x))))^*(\delta^* \mu(\phi, \psi))).$$

Strictly speaking, we must also compose $\delta^* \mu(\phi, \psi)$ with the appropriate coherence maps and the (unique) iso which is part of the equivalence $\mu$, but the notation is already heavy, so we leave those implicit.

Now $(z, \iota_X \pi_X((\langle f, g \rangle z, x)) = Z \times \iota_X(x) : Z \times X \rightarrow Z \times (X + Y)$; by (1), and again keeping the coherence maps that are part of the composition implicit, we get,

$$(Z \times \iota_X)^*(\delta^* \mu(\phi, \psi)) = \iota_{Z \times X}^*(\mu(\phi, \psi)) = \phi,$$

moreover

$$\text{id}_{\alpha(\pi_U(u, v), x)}((f, g) z, x) = \text{id}_{\alpha(\pi_U(fz, gz), x)} = \text{id}_{\alpha(fz, x)},$$

since reindexing is functorial, so the composite is $(f, F, \phi)$ as needed. Uniqueness is clear by inspection.

The terminal object of $\text{Dial}(p)$ is $(0 \rightarrow 1)$, where is the unique object of $\mathcal{E}(0)$. The product functor works as follows: Given

$$(f, F, \phi(u, x')) : A = (U \xleftarrow{\alpha} X) \rightarrow A' = (U' \xleftarrow{\alpha'} X')$$

$$(g, G, \psi(v, y')) : B = (V \xleftarrow{\beta} Y) \rightarrow B' = (V' \xleftarrow{\beta'} Y')$$

The product $(f, F, \phi(u, x')) \times (g, G, \psi(v, y')) : A \times B \rightarrow A' \times B'$ is

$$(f \times g, F(\pi_U(u, v), x') + G(\pi_V(u, v), y'), \delta^* \mu[\phi(\pi_U(u, v), x'), \psi(\pi_V(u, v), y')].$$
Example 3.3. Examples of fibrations satisfying Proposition 3.2 (which are actually equivalent to codomain fibrations) are the split fibrations Fam(Set) \to Set. For set-indexed families of sets we have \mu((A_i)_{i \in I}, (B_i)_{i \in I}) = (C_z)_{z \in X+Y}, where
\[
C_z = \begin{cases} 
A_x & \text{if } z = (0, x) \\
B_y & \text{if } z = (1, y)
\end{cases}
\]

and UFam(PER) \to PER (for a description of this fibration, see Section 5). For per-indexed families of pers, \mu((A_{[n]})_{[n] \in N/R}, (B_{[m]})_{[m] \in N/S}) = (C_{[k]})_{[k] \in N/R+S}, where
\[
(C_{[k]})_{[k] \in N/R+S} = \begin{cases} 
A_{[n]} & \text{if } pk = 0 \text{ and } [p/k] = [n] \\
B_{[m]} & \text{if } pk = 1 \text{ and } [p/k] = [m]
\end{cases}
\]

We now show under which conditions our case study, the codomain fibration, satisfies the assumptions of Proposition 3.2.

Definition 3.4. A category is said to have stable coproducts, if for \( A = \coprod_{i \in I} A_i \) and any map \( f : B \to A \), \( \coprod_{i \in I} f^{-1}(A_i) \cong B \), where \( f^{-1}(A_i) \) is the pullback of the injection \( A_i \to \coprod_{i \in I} A_i \) along \( f \).

Definition 3.5. A category is said to have disjoint coproducts if each of the injections \( \iota_i : A_i \to A \) is mono and for each pair of distinct injections \( \iota_i, \iota_j \), the pullback of the two is the initial object.

The following fact is well known (see, e.g., [CLW93])

Proposition 3.6. A category that has finite limits and finite, stable, disjoint coproducts is distributive.

Now let \( C \) be a category with finite products and coproducts, and assume that the coproducts are stable and disjoint, then \( \text{cod}(C) \) is a cloven fibration with an equivalence \( \mu : C/X \times C/Y \to C/(X+Y) \) given by \( \mu(\alpha, \beta) = \alpha + \beta \) and \( \mu^{-1}(\gamma) = (\iota_X^*(\gamma), \iota_Y^*(\gamma)) \). This means that \( \text{Dial}(\text{cod}(C)) \) has products.

We sum up the results in the following proposition:

Proposition 3.7. Let \( C \) be a category with finite limits and finite coproducts, and assume that the coproducts are stable and disjoint, then \( \text{Dial}(\text{cod}(C)) \) has finite products.
3 DIALECTICA CATEGORIES FOR CLOVEN FIBRATIONS

3.1 Products in Dial(p)

Using the internal language, this means that

\[ \hat{\alpha}(u,x) = \begin{cases} \alpha(u,x) & \text{if } x \in X \\ \top & \text{if } x \in 1. \end{cases} \]

And

\[ L^+(f,F) = (f, \hat{F}), \]

where \( \hat{F} \) is the composite

\[ U \times (Y + 1) \xrightarrow{(U \times Y) + !Y} (U \times Y + 1) \xrightarrow{F + 1} X + 1 \]

In the internal language this becomes

\[ \hat{F}(u,y) = \begin{cases} F(u,y) & \text{if } y \in Y \\ \ast \in 1 & \text{if } y \in 1. \end{cases} \]

Now, it is the Kleisli category Dial_{L^+} for this comonad that we are really interested in.

In the case of the free commutative monoid monad, there is an isomorphism

\[ X^* \times Y^* \cong (X + Y)^* \] (2)

which induces an isomorphism

\[ !(A \times B) \cong !A \otimes !B \] (3)

in Dial(Sub(C)). A comonad \( L \) satisfying \( L(A \times B) \cong LA \otimes LB \) will be called Girardian. If a comonad is Girardian, the isomorphism in 3 gives us a Cartesian closed structure on the Kleisli category by the following string of equivalences:

\[ \text{Hom}_{Dial}(A \times B,C) \cong \text{Hom}_{Dial}(!(A \times B),C) \cong \text{Hom}_{Dial}(!(A \otimes B),C) \cong \text{Hom}_{Dial}(!A,[B,C]_{Dial}) = \text{Hom}_{Dial}(A,[B,C]_{Dial}) \] (4)

Now, for the monad \( - + 1 \) we do not have such an isomorphism, because

\[ (X + 1) \times (Y + 1) \cong X + Y + 1 + (X \times Y) \neq X + Y + 1 \]

So the monad \( - + 1 \) does not satisfy the distributive law in 2, and one readily sees that the comonad \( L^+ \) does not satisfy the distributive law in 3. However, we shall see that what we do have is a natural retraction

\[ \text{Hom}_{Dial}(L^+(A \times B),C) \rightarrow \text{Hom}_{Dial}(L^+(A),B \supset C), \]

so \( B \supset C \) is the weak exponent, that we will define in the next section. Notice that \( B \supset C \) is not simply \([LB,C]_{Dial} \). Hence for the Kleisli category Dial_{L^+}, we will have a natural retraction

\[ \text{Hom}_{Dial_{L^+}}(A \times B,C) \rightarrow \text{Hom}_{Dial_{L^+}}(A,B \supset C). \]

And then the Cauchy completion (see Appendix A) will give a Cartesian closed category.

**Definition 3.8** (Weak exponential). Let \( C \) be a category with finite products. \( C \) has weak exponentials \([B,C]\), if there is a retraction

\[ C(A \times B,C) \xrightarrow{1} C(A,[B,C]) \]

onto \( C(A \times B,C) \) (that is, \( RI = id \)), natural in \( A \).
4 A non-Girardian comonad on Dial(p)

In this section we study which conditions are needed on the monad and the fibration to give a well-defined comonad on Dial(p).

Recall the definition of a strong monad on a category with finite products.

**Definition 4.1.** Let $\mathcal{C}$ be a category with finite products. A strong monad on $\mathcal{C}$ is a monad $(T, \eta, \mu)$ together with a natural transformation $C_{X,Y} : X \times TY \to T(X \times Y)$ called strength, such that the diagrams

\[
\begin{align*}
X \times Y &\xrightarrow{X \times \eta_Y} X \times TY \xrightarrow{C_{X,Y'}} T(X \times Y), \\
&\downarrow \eta_{X \times Y} \\
T(X \times Y) &\phantom{\xrightarrow{T(X \times Y)}}
\end{align*}
\]

\[
\begin{align*}
X \times TY &\xrightarrow{C_{X,Y}} T(X \times Y) \xrightarrow{T(\mu_{X,Y})} T^2(X \times Y), \\
&\downarrow \mu_{X,Y} \\
T(X \times Y) &\phantom{\xrightarrow{T(X \times Y)}}
\end{align*}
\]

\[
\begin{align*}
(X \times Y) \times TZ &\xrightarrow{C_{X \times Y,Z}} T((X \times Y) \times Z), \\
&\cong \\
X \times (Y \times TZ) &\xrightarrow{C_{X,Y \times Z}} X \times T(Y \times Z) \xrightarrow{\mu_{X,Y \times Z}} T(X \times (Y \times Z)) \\
&\cong
\end{align*}
\]

commute for all objects $X, Y$ and $Z$.

We aim to use a monad to define a comonad on Dial(p) for $p$ a cloven fibration, for that we require the monad to be fibred. A fibred monad $(T, T')$ on a fibration $p : \mathcal{E} \to T$ is a morphism of fibrations $(T, T')$:

\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{T} & \mathcal{E} \\
\downarrow p & & \downarrow p \\
T & \xrightarrow{T'} & T
\end{array}
\]

Together with 2-cells $\mu = (\mu, \mu')$ and $\eta = (\eta, \eta')$:

\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{T^2} & \mathcal{E} \\
\downarrow \mu & & \downarrow \mu' \\
T & \xrightarrow{(T')^2} & T
\end{array}
\]

where $\eta$ is above $\eta'$, that is, for $X \in \mathcal{E}$, $p(\eta_X) = \eta'_pX$, and similar for $\mu$. $(T, T')$ commutes with reindexing in the sense that for $u : X \to Y$ in $\mathcal{T}$, we have $T \circ u^* \cong (T'u)^* \circ T$ as fibred functors from $\mathcal{E}(Y) \to \mathcal{E}(T'X)$. For more details, see e.g. [Jac99].

Suppose we have a fibred monad $(T, T')$, where $T'$ is strong, then we are able to define a comonad $L$ on Dial(p) by
• $L(U, X, \alpha) = (U, T'X, C_{U,X}^*(T\alpha) = \hat{\alpha})$ and

• $L(f, F, \phi) = (f, T'(F) \circ C_{U,Y}, C_{U,Y}^*(T\phi)) = (f, \hat{F}, \hat{\phi})$.

For the arrow part of the functor $L$ to be well-defined we must show that $C_{U,Y}^* T$ commutes with reindexing, that is,

$$C_{U,Y}^* T(\alpha(u, F(u, y))) = (C_{U,Y}^* T(\alpha))(u, \hat{F}(u, y)) = \hat{\alpha}(u, \hat{F}(u, y)).$$  \hspace{1cm} (5)

and

$$C_{U,Y}^* T(\beta(fu, y)) = (C_{U,Y}^* T(\beta))(fu, y) = \hat{\beta}(fu, y).$$  \hspace{1cm} (6)

Equation (6) holds because $(T, T')$ is a morphism of fibrations and therefore commutes with reindexing, and because of naturality of $C$:

$$C_{U,Y}^* T(\beta(fu, y)) = C_{U,Y}^* T((f \times Y)^* \beta) \cong C_{U,Y}^* T((f \times Y)^*(T(\alpha))) \cong (f \times T'Y)^* C_{U,Y}^*(T\beta) = \hat{\beta}(fu, y).$$

To see that (5) holds, consider

$$C_{U,Y}^* T(\alpha(u, F(u, y))) = C_{U,Y}^* T(\langle \pi_U, F \rangle^*(\alpha)) \cong C_{U,Y}^* T(\langle \pi_U, F \rangle^*(T(\alpha))) \cong (T(\langle \pi_U, F \rangle) \circ C_{U,Y}^*(T\alpha)) \cong (C_{U,X} \circ \langle \pi_U, T'(F) \circ C_{U,Y} \rangle)^*(T\alpha) \cong \hat{\alpha}(u, \hat{F}(u, y)).$$

We must show that the following diagram commutes in $T$:

\[
\begin{array}{ccc}
U \times T'Y & \xrightarrow{C_{U,Y}} & T'(U \times Y) \\
\langle \pi_U, T'F \circ C_{U,Y} \rangle \downarrow & & \downarrow T'(\pi_U, F) \\
U \times T'X & \xleftarrow{C_{U,X}} & T'(U \times X)
\end{array}
\hspace{1cm} (7)
\]

The diagram (7) can be decomposed as:

\[
\begin{array}{ccc}
U \times T'Y & \xrightarrow{C_{U,Y}} & T'(U \times Y) \\
\delta_U \times T'Y \downarrow & & \downarrow T'(\delta_U \times Y) \\
U \times U \times T'Y & \xrightarrow{C_{U \times U,Y}} & T'(U \times U \times Y) \\
U \times C_{U,Y} \downarrow \hspace{1cm} (1) & & \hspace{1cm} \downarrow \text{id} \\
U \times T'(U \times Y) & \xrightarrow{C_{U \times U,Y}} & T'(U \times U \times Y) \\
U \times T'F \downarrow \hspace{1cm} (2) & & \hspace{1cm} \downarrow T'(U \times F) \\
U \times T'X & \xrightarrow{C_{U,X}} & T'(U \times X)
\end{array}
\]

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where (1) and (3) commute by naturality of $C$, and (2) commutes by properties of strength of $T'$.

Assume the following extra requirements on $(T, \eta, \mu)$:

$$\begin{align*}
\alpha(u, x) &= \eta_{U \times X}^*(T\alpha) \\
T^2(\alpha) &= \mu_{U \times X}^*(T\alpha)
\end{align*}$$

(8)

Using the properties of strength one can show that the equations in (8) imply the following equations:

$$\begin{align*}
\hat{\alpha}(u, \eta_X(x)) &= \alpha(u, x) & \text{and} \\
\hat{\alpha}(u, \mu_X(x')) &= \hat{\alpha}(u, x').
\end{align*}$$

(9)

We now use these assumptions to show that $L$ is a comonad:

$L$ is a comonad on $\text{Dial}(p)$: For every $(U, X, \alpha)$ we have a map

$$
\begin{array}{c}
U \\
\downarrow \alpha \\
X
\end{array}
\begin{array}{c}
\downarrow \hat{\alpha} \\
TX
\end{array}
\begin{array}{c}
\downarrow \eta_X \pi_X \\
\downarrow \id \\
U
\end{array}
\begin{array}{c}
\downarrow \id \\
\downarrow \alpha
\end{array}
\begin{array}{c}
\downarrow \id \\
\downarrow \hat{\alpha}
\end{array}
\begin{array}{c}
\downarrow \id \\
\downarrow \hat{\alpha}
\end{array}
\begin{array}{c}
\downarrow \id \\
\downarrow \id
\end{array}
\begin{array}{c}
\downarrow \id \\
\downarrow \id
\end{array}
\begin{array}{c}
\downarrow \id \\
\downarrow \id
\end{array}
$$

with

$$\hat{\alpha}(u, \eta_X(x)) = \alpha(u, x) \quad \text{and} \quad \hat{\alpha}(u, \mu_X(x')) = \hat{\alpha}(u, x').$$

since from (9) we have $\hat{\alpha}(u, \eta(x)) = \alpha(u, x)$. We define

$$\varepsilon_{(U, X, \alpha)} = (\id_U, \eta_X \pi_X, \id_\alpha).$$

And for every $(U, X, \alpha)$ we have a map

$$
\begin{array}{c}
U \\
\downarrow \alpha \\
X
\end{array}
\begin{array}{c}
\downarrow \hat{\alpha} \\
TX
\end{array}
\begin{array}{c}
\downarrow \mu_X \pi \\
\downarrow \id \\
U
\end{array}
\begin{array}{c}
\downarrow \id \\
\downarrow \alpha
\end{array}
\begin{array}{c}
\downarrow \id \\
\downarrow \hat{\alpha}
\end{array}
\begin{array}{c}
\downarrow \id \\
\downarrow \hat{\alpha}
\end{array}
\begin{array}{c}
\downarrow \id \\
\downarrow \id
\end{array}
\begin{array}{c}
\downarrow \id \\
\downarrow \id
\end{array}
\begin{array}{c}
\downarrow \id \\
\downarrow \id
\end{array}
\begin{array}{c}
\downarrow \id \\
\downarrow \id
\end{array}
$$

with

$$\hat{\alpha}(u, \mu_X(x')) = \hat{\alpha}(u, x') \quad \text{and} \quad \hat{\alpha}(u, \mu_X(x')) = \hat{\alpha}(u, x').$$

We have shown the following.

**Proposition 4.2.** Let $((T, T'), (\eta, \eta'), (\mu, \mu'))$ be a fibred monad with $T'$ a strong monad on a cloven fibration $p : E \to T$, and with $T$ ccc. If $(T, \eta, \mu)$ also satisfies the equations in (8), we can define a comonad $L$ on $\text{Dial}(p)$ by

$L(U, X, \alpha) = (U, TX, C^*_U X (T\alpha) = \hat{\alpha})$ and $L(f, F, \phi) = (f, TF \circ C_U Y, C^*_U Y (T(\phi))) = (f, \hat{F}, \hat{\phi})$. 

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The Comonad $L^+$  Our leading example is the comonad $L^+$ on $\text{Dial}(\text{cod}(C))$ based on the monad $TX = X + 1$, which is strong and induces a fibred monad on $\text{cod}(C)$. We have

**Lemma 4.3.** For a Cartesian closed category $C$ with finite coproducts, the functor $TX = X + 1$ together with the obvious natural transformations $\mu_X : X + 1 + 1 \to X + 1$ and $\eta_X : X \to X + 1$ is a strong monad. The maps $C_{X,Y}$ are defined by

$$X \times (Y + 1) \cong X \times Y + X \xrightarrow{X \times Y + !} X \times Y + 1$$

Assuming $C$ has stable, disjoint coproducts, it is not hard to see that the equations in (8) are met by the monad $- + 1$. We collect the facts:

**Proposition 4.4.** Suppose $C$ has finite limits and coproducts, and that coproducts are stable and disjoint. Then the monad $- + 1$ gives rise to a comonad on $\text{Dial}(\text{cod}(C))$.

Some examples:

**Example 4.5.** Examples that satisfy Proposition 4.2 are the codomain fibration together with the monads $- + 1$ and strings; and the subobject fibration together with the monads multisets, powersets, finite powersets and the free commutative monoid monad. The Kleisli category for the latter is the Diller-Nahm Dialectica category. The comonad based on $- + 1$ is the only non-Girardian among these examples.

### 4.1 The Kleisli Category $\text{Dial}_L(p)$

We now write out some details about the category $\text{Dial}_L(p)$ for a comonad $L$ on $\text{Dial}(p)$.

$\text{Dial}_L$ has products inherited from $\text{Dial}$. For the record:

**Proposition 4.6.** The Kleisli category $\text{Dial}_L(p)$ has products inherited from $\text{Dial}(p)$.

**Proof:**

$$\text{Dial}_L(\alpha, \beta) \times \text{Dial}_L(\alpha, \gamma) = \text{Dial}(\hat{\alpha}, \beta) \times \text{Dial}(\hat{\alpha}, \gamma)$$

$$\cong \text{Dial}(\hat{\alpha}, \beta \times \gamma)$$

$$= \text{Dial}_L(\alpha, \beta \times \gamma).$$

Composition in the Kleisli category $\text{Dial}_L(p)$  Given two maps $(f, F, \phi(u, y)) : (U, TX, \hat{\alpha}) \to (V, Y, \beta)$ and $(g, G, \psi(v, w)) : (V, TY, \hat{\beta}) \to (Z, W, \gamma)$ the composite is

\[
\begin{array}{ccc}
U & \xrightarrow{\hat{\alpha}} & TX \\
\downarrow g \downarrow f & & \downarrow \mu_X \circ \hat{F}(u, G(fu, w)) \\
Z & \xleftarrow{\hat{\beta}} & W
\end{array}
\]

with

\[
\hat{\alpha}(u, \mu_X \circ \hat{F}(u, G(fu, w))) = \hat{\alpha}(u, \hat{F}(u, G(fu, w))) \xrightarrow{\hat{\phi}(u, G(fu, w))} \hat{\beta}(fu, G(fu, w)) \xrightarrow{\psi(fu, w)} \gamma(gfu, w).
\]
4 A NON-GIRARDIAN COMONAD ON Dial(P)

## 4.2 The Kleisli category Dial\(^+\) has a weak exponent.

### Product functor in the Kleisli category Dial\(_L(p)\)

Given maps

\[
\begin{array}{ccc}
U' & \xrightarrow{f} & TX' \\
\downarrow{\phi} & & \downarrow{\alpha}\circ F \\
U & \xrightarrow{\alpha} & X
\end{array}
\]

and

\[
\begin{array}{ccc}
V' & \xrightarrow{g} & TY' \\
\downarrow{\psi} & & \downarrow{\alpha}\circ G \\
V & \xrightarrow{\beta} & Y
\end{array}
\]

The product is

\[
\begin{array}{ccc}
U' \times V' & \xrightarrow{\alpha' \& \beta'} & T(X' + Y') \\
\downarrow{\phi \circ \pi + \psi \circ \pi} & & \downarrow{\alpha'(u', v') + \beta'(v', y)} \\
U \times V & \xrightarrow{\alpha \& \beta} & X + Y
\end{array}
\]

For a Girardian comonad, the Kleisli category is automatically Cartesian closed (see (4)). We now show that for the non-Girardian comonad \(L^+\) constructed from \((- + 1\)), we can define a weak exponent in the Kleisli category. Notice that the weak exponent is not simply the usual \([L^+ A, B]_{\text{Dial}}\).

### Theorem 4.7

Let \(\mathcal{C}\) be a ccc with finite limits, and stable, disjoint coproducts, and which is locally Cartesian closed. We have already seen that the monad \((- + 1\)) satisfies the requirements needed to construct a comonad \(L^+\) on Dial(cod(\(\mathcal{C}\))). Let Dial\(^+\) denote the Kleisli category for \(L^+\) on Dial(cod(\(\mathcal{C}\))). We are going to show that Dial\(^+\) has weak exponentials. That is,

**Theorem 4.7.** Let \(\mathcal{C}\) be a ccc with finite limits, and stable, disjoint coproducts, and which is locally Cartesian closed, then the Dialectica-Kleisli category, Dial\(_L^+(\text{cod}(\mathcal{C}))\), which we denote by Dial\(^+\), and also Dial\(_L^+(\text{Sub}(\mathcal{C}))\) has finite products and weak exponentials.

First some notation that we shall be using.

12
Notation Let $C$ be a ccc with stable, disjoint coproducts and consider a pullback of the form:

\[
\begin{array}{c}
F^{-1}(\alpha + \beta) \\
\downarrow \quad a+b
\end{array} \xleftarrow{\alpha + \beta} \quad \begin{array}{c}
X + Y \\
\downarrow \quad \alpha + \beta
\end{array}
\]

Since we have stable, disjoint coproducts, we have

\[
U \times V \simeq F^{-1}(X) + F^{-1}(Y)
\]

and pullback preserves coproducts, so

\[
F^{-1}(\alpha + \beta) \simeq F^{-1}(\alpha) + F^{-1}(\beta)
\]

which, because of the stable, disjoint coproducts is the same as

\[
F_{X}^{-1}(\alpha) + F_{Y}^{-1}(\beta)
\]

Note that this holds in any fibration over $C$ where reindexing preserves coproducts.

We will use the following notation for $F_{X}^{-1}(\alpha) + F_{Y}^{-1}(\beta)$ in this situation:

\[
F(u, v) \in X.\alpha(F(u, v)) + F(u, v) \in Y.\beta(F(u, v))
\]

or sometimes, when convenient:

\[
F(u, v) = x \in X.\alpha(x) + F(u, v) = y \in Y.\beta(y)
\]

indicating that $\alpha$ is being reindexed along those $(u, v)$ such that $F(u, v) \in X$ and $\beta$ along those $(u, v)$ such that $F(u, v) \in Y$.

In case $\alpha$ (or $\beta$) is the terminal object of the fibre, in this case if $\alpha = X, a = id$, the pullback will be the type

\[
F(u, v) \in X.\top_{X}(F(u, v)) + F(u, v) \in Y.\beta(F(u, v))
\]

By abuse of notation we will sometimes leave out the first part and just write this type as

\[
F(u, v) \in Y.\beta(F(u, v)).
\]

We are now able to give the proof of the Theorem 4.7:

Proof. First we define an object corresponding to

\[
\square = \{ (g,G) : (V \Rightarrow W) \times (V \times Z \Rightarrow 1 + Y), (v,z) : V \times Z,
\]

\[
k(g,G,v,z) : G(v,z) = y \in Y.[\beta(v,G(v,z)),\gamma(gv,z)]\}
\]

where $[\beta,\gamma]$ is the fibred exponential, and $G(v,z) = y \in Y.[\beta(v,G(v,z)),\gamma(gv,z)]$ means (informally) that we reindex the fibred exponent $[\beta,\gamma]$ along those $(G,g,v,z)$ such that
$G(v, z) \in Y$, which makes sense since we have disjoint, stable coproducts. In our case study, the codomain fibration, this is the dependent type defined by the pullback:

\[
\begin{array}{ccc}
\square & \rightarrow & W \times V \times Z + [W \times Z \times \beta, V \times Y \times \gamma] \\
\downarrow & & \downarrow \text{id} + b \\
(V \Rightarrow W) \times (V \times Z \Rightarrow 1 + Y) \times V \times Z & \rightarrow & W \times V \times Z + W \times V \times Z \times Y
\end{array}
\]

where $(\pi \Rightarrow W) \times \text{id} : (V \Rightarrow W) \times (V \times Z \Rightarrow 1 + Y) \times V \times Z \rightarrow (V \times Z \Rightarrow W) \times (V \times Z \Rightarrow 1 + Y) \times V \times Z$. Since $B$ is lcc we have a right adjoint to reindexing $\pi^* \dashv \Pi_\pi$. Let $\pi : (V \Rightarrow W) \times (V \times Z \Rightarrow 1 + Y) \times V \times Z \rightarrow (V \Rightarrow W) \times (V \times Z \Rightarrow 1 + Y)$ then we have

\[
(\Pi_\pi \square) \times V \times Z \xrightarrow{\varepsilon} \square \quad (V \Rightarrow W) \times (V \times Z \Rightarrow 1 + Y) \times V \times Z
\]

where $\varepsilon$ is the counit for the adjunction $\pi^* \dashv \Pi_\pi$. We have $\Pi_\pi \square = \{(g, G) : (V \Rightarrow W) \times (V \times Z \Rightarrow 1 + Y), \lambda v, z. K(g, G, v, z) : \Pi_\pi, v, z. G(v, z) = y \in Y, [\beta(v, G(v, z)), \gamma(g(v), z)]\}$

Now the exponent in Dial$^+$, which we will denote $\beta \supset \gamma$, is the dependent type over $(\Pi_\pi \square) \times V \times Z$ corresponding to

\[
(\beta \supset \gamma)((g, G, k), (v, z)) := G(v, z) \in 1.\gamma(g(v), z)
\]

defined by the pullback:

\[
\begin{array}{ccc}
\beta \supset \gamma & \rightarrow & (\gamma \times V) + [W \times Z \times \beta, V \times Y \times \gamma] \\
\downarrow & & \downarrow \text{c+id} \\
(\Pi_\pi \square) \times V \times Z & \xrightarrow{\varepsilon} & W \times V \times Z + [W \times Z \times \beta, V \times Y \times \gamma]
\end{array}
\]

In the type theoretic language this is

$\beta \supset \gamma = \{(g, G, \lambda v, z. k(g, G, v, z)) : \Pi_\pi \square, v, z \in V \times Z, h(g, G, \lambda v, z. k, v, z) : G(v, z) \in 1.\gamma(g(v), z)\}$

Before we show that there is a natural retraction

\[
\text{Dial}^+(\alpha \times \beta, \gamma) \xrightarrow{\text{R}} \text{Dial}^+(\alpha, \beta \supset \gamma),
\]

we will characterize the homsets. A map from $\alpha \& \beta$ to $\gamma$ in Dial$^+$:

\[
\begin{array}{ccc}
U \times V & \xrightarrow{\alpha \& \beta} & X + 1 \\
\downarrow f & & \downarrow \phi(u, v, z) : \alpha \& \beta(u, v, F(u, v, z)) \rightarrow \gamma(f(u, v), z) \\
W & \xrightarrow{\gamma} & Z
\end{array}
\]
Now because we have stable and disjoint coproducts, we can write \( \phi(u, v, z) \) as

\[
\phi(u, v, z) = \phi_X(u, v, z) + \phi_Y(u, v, z) + \phi_1(u, v, z)
\]

where the maps \( \phi_X, \phi_Y, \phi_1 \) are the results of pulling back \( \phi \) along respectively \( F^{-1}(X) \to U \times V \times Z \), \( F^{-1}(Y) \to U \times V \times Z \), and \( F^{-1}(Z) \to U \times V \times Z \).

A map from \( \alpha \) to \( \beta \circ \gamma \) in \( \text{Dial}^+ \):

\[
\begin{array}{ccc}
\psi(u, v, z) : \alpha(u, K(u, v, z)) \to (\beta \circ \gamma)(f(u), H(u), k(f(u), H(u)), v, z) \\
\end{array}
\]

Again we use the coproduct properties to get:

\[
\psi(u, v, z) = \psi_X(u, v, z) + \psi_1(u, v, z)
\]

Spelling out what this means:

\[
\hat{\alpha}(u, K(u, v, z)) \to (\beta \circ \gamma)(f(u), H(u), k(f(u), H(u)), v, z) \\
\gamma \times V + [\beta, \gamma]
\]

where the square is a pullback. Notice that \( U \times V \times Z \cong H^{-1}(1) + H^{-1}(Y) \) and

\[
(\beta \circ \gamma)(f, H, k(f, H), v, z) \cong \gamma(f(u, v), z) + H^{-1}(Y),
\]

so the triangle in the diagram can be written

\[
\hat{\alpha}(u, K(u, v, z)) \to \gamma(f(u, v), z) + H^{-1}(Y)
\]

Since pullbacks preserves coproducts and

\[
U \times V \times Z \cong (K^{-1}(X) \wedge H^{-1}(1)) + (K^{-1}(X) \wedge H^{-1}(Y)) + (K^{-1}(1) \wedge H^{-1}(1)) + (K^{-1}(1) \wedge H^{-1}(Y)).
\]
where $K^{-1}(1) \land H^{-1}(1)$ means pullback of $K^{-1}(1) \to U \times V \times Z$ and $H^{-1}(1) \to U \times V \times Z$.

We then get

\[
\psi_X = \psi_{X,1} : K(u,v,z) = x \in X \land H(u,v,z) = * \in 1. \quad \alpha(u,x) \to \gamma(f(u,v),z)
\]

\[
+ \psi_{X,Y} : K(u,v,z) = x \in X \land H(u,v,z) = y \in Y. \quad \alpha(u,x) \to \top
\]

\[
\psi_1 = \psi_{1,1} : K(u,v,z) = * \in 1 \land H(u,v,z) = * \in 1. \quad \top \to \gamma(f(u,v),z)
\]

\[
+ \psi_{1,Y} : K(u,v,z) = * \in 1 \land H(u,v,z) = y \in Y. \quad \top \to \top
\]

So when $H(u,v,z) \in Y$ we get no information from $\psi$, however,

\[
(f,H,k(f,H)) : U \times V \times Z \to \square
\]

and in particular, we have, in the fibre over $H^{-1}(Y)$,

\[
\tilde{k}(f(u),H(u),v,z) : H(u,v,z) = y \in Y.[\beta(v,H(u,v,z),\gamma(f(u,v),z))].
\]

Now we are ready to give the retraction.

\[
I : \text{Dial}^+(\alpha \& \beta, \gamma) \to \text{Dial}^+(\alpha, \beta \triangleright \gamma)
\]

$I$ works as follows. Given $f,F,\phi$ we get

- $\hat{f}$, the transposed of $f$ by the Cartesian closure of $\mathcal{C}$.
- $H = U \times V \times Z \xrightarrow{F} X + Y + 1 \xrightarrow{-} Y + 1$,
- $\tilde{k}(\hat{f}(u),H(u),v,z) = \phi_Y(u,v,z) : F(u,v,z) = H(u,v,z) = y \in Y. \beta(v,y) \to \gamma(f(u,v),z)$,
- $K = U \times V \times Z \xrightarrow{F} X + Y + 1 \xrightarrow{-} X + 1$
- $\psi(u,v,z) = \\
\phi_X(u,v,z) : K^{-1}(X) \land H^{-1}(1) = F^{-1}(X). \quad \alpha(u,F(u,v,z)) \to \gamma(f(u,v),z)$
- $\phi_1(u,v,z) : K^{-1}(1) \land H^{-1}(1) = F^{-1}(1). \quad \top \to \gamma(f(u,v),z)$
- $\text{id} : K^{-1}(1) \land H^{-1}(Y) = F^{-1}(Y). \quad \top \to \top$

Note that $K^{-1}(X) \land H^{-1}(Y) = 0$ since $F^{-1}(X) \land F^{-1}(Y) = 0$.

\[
R : \text{Dial}^+(\alpha, \beta \triangleright \gamma) \to \text{Dial}^+(\alpha \& \beta, \gamma)
\]

Given $(f,H,k(f,H)), K, \psi, R$ returns

- $\hat{f}$, the transpose of $f$.
- $F = H \upharpoonright_{H^{-1}(Y)} + K \upharpoonright_{H^{-1}(1)}$
- $\phi(u,v,z) : \alpha \& \beta(u,v,F(u,v,z)) \to \gamma(f(u,v),z)$ is defined by
\[
\phi(u,v,z) = \\
\psi_{X,1} : F^{-1}(X) = K^{-1}(X) \land H^{-1}(1). \quad \alpha(u,F(u,v,z)) \to \gamma(f(u,v),z)
\]
\[
+ \psi_{1,1} : F^{-1}(1) = K^{-1}(1) \land H^{-1}(1). \quad \top \to \gamma(f(u,v),z)
\]
\[
+ k(f(u),h(u),v,z) : F(u,v,z) = H(u,v,z) = y \in Y. \quad \beta(v,y) \to \gamma(f(u,v),z)
\]

which is the same as saying $\phi = \psi \upharpoonright_{H^{-1}(1)} + k$.

It is now straightforward to verify that $RI = \text{id}$.
4 A NON-GIRARDIAN COMONAD ON Dial(P)

4.2 The Kleisli category Dial⁺ has a weak exponent.

Naturality of the retraction  Let

\[
\begin{array}{ccc}
U' & \xrightarrow{\hat{\alpha}'} & X' + 1 \\
& \downarrow T & \\
& & \theta(u', x) : \hat{\alpha}'(u, v, F(u', T(u', x))) \rightarrow \alpha(t(u'), x)
\end{array}
\]

and \((f, F, \phi) : \alpha \& \beta \rightarrow \gamma,\) and \((f, H, k, K, \psi) : \hat{\alpha} \rightarrow (\beta \supset \gamma).\) We know that \(\text{id}_\beta = (\text{id}_V, \mu_Y \circ \pi_Y, \text{id}_\beta)\) and

\[
(t, T, \theta) \& \text{id}_\beta = (t \times \text{id}_V, \mu_X \circ (T + \pi_Y), \theta(\pi(u', v'), x) + \text{id}_\beta)
\]

We must show that

\[
I(f, F, \phi) \circ (t, T, \theta) = \alpha(t(u'), x)
\]

and

\[
I((f, H, k, K, \psi) \circ (t, T, \psi)) = R((f, H, k, K, \psi) \circ ((t, T, \theta) \& \text{id}_\beta))
\]

For (10) consider

\[
I(f, F, \phi) = (f, H, k = \phi_Y, K, \psi = \phi_X + \phi_1 + \text{id})
\]

so the left hand side of (10) becomes

\[
I(f, F, \phi) \circ (t, T, \theta) =
\]

- \((f, H, \phi_Y) \circ t = (f(t(u'), v), H(t(u'), v, z), \phi_Y(t(u'))) : U' \rightarrow \Pi \square,\)
- \(\hat{T}(u', K(t(u'), v, z)) : U' \times V \times Z \rightarrow X' + 1,\)
- the composite

\[
\begin{array}{ccc}
\hat{\alpha}'(u', \hat{T}(u', K(t(u'), v, z))) & \xrightarrow{\hat{\theta}(u', K(t(u'), v, z))} & \hat{\alpha}(t(u'), K(t(u'), v, z)) \\
& \downarrow \psi(t(u'), v, z) & \\
& (\beta \supset \gamma)(f(t(u')), H(t(u')), \phi_Y(t(u')), v, z).
\end{array}
\]

On the other hand

\[
I((f, F, \phi) \circ (t, T, \theta) \& \text{id}_\beta) =
\]

\[
I[f(t(u'), v), (T + \pi_Y)(u', v, F(t(u'), v, z)), (\phi(t(u'), v, z) \circ (\theta + \text{id}_\beta))(u', v, F(t(u', v, z)))]
\]

which yields

- \(f(t(u'), v),\)
\[ H = U' \times V \times Z^{(u', v, F(t(u'), v, z))} \rightarrow U' \times V \times (X + Y + 1) \xrightarrow{(T + \pi_Y)} X' + Y + 1 \rightarrow Y + 1 \]

\[ k(f(t(u')), H(t(u')), v, z) = \]

\[ \phi_Y(t(u'), v, z) : F(t(u'), v, z) = y \in Y. \hat{\beta}(v, y) \xrightarrow{id} \hat{\beta}(v, y) \xrightarrow{\phi_Y(t(u'), v, z)} \gamma(t(u'), v, z) \]

\[ K = U' \times V \times Z^{(u', v, F(t(u'), v, z))} \rightarrow U' \times V \times (X + Y + 1) \xrightarrow{(T + \pi_Y)} X' + Y + 1 \rightarrow X' + 1 \]

\[ \psi = \]

\[ \phi_X(t(u'), v, z) \circ \hat{\theta}(u', x) : F(t(u'), v, z) = x \in X. \hat{\alpha}'(u', T(u', x)) \rightarrow \hat{\alpha}(t(u'), x) \rightarrow \gamma(t(u'), v, z) \]

\[ + \phi(t(u'), v, z) : F(t(u'), v, z) \in T \rightarrow \gamma(t(u'), v, z) \]

\[ + \]

\[ \text{id} : F(t(u'), v, z) = y \in Y. T \rightarrow T. \]

which is easily seen to be equal.

To see that (11) holds, recall that

\[ R(f, H, k, K, \psi) = (f, F = H \upharpoonright H^{-1}(Y) + K \upharpoonright H^{-1}(1), \psi \upharpoonright H^{-1}(1) + k) \]

Now, the right hand side of 11 is

\[ f(t(u'), v), \]

\[ \bar{T} + \pi_Y(u', v, F(t(u'), v, z)), \]

\[ \]

\[ \psi \upharpoonright H^{-1}(1) (t(u'), v, z) + k(t(u')) \circ \hat{\theta} + \text{id}_\beta(u', v, F(t(u'), v, z)) \]

\[ = \psi \upharpoonright H^{-1}(1) (t(u'), v, z) + k(t(u')) \circ \hat{\theta}(u', v, F_{X+1}(t(u'), v, z)) + \text{id}_\beta(v, F_Y(t(u'), v, z))) \]

\[ = \psi \upharpoonright H^{-1}(1) (t(u'), v, z) + k(t(u')) \circ \hat{\theta}(u', K \upharpoonright H^{-1}(1) (t(u'), v, z)) + \text{id}_\beta(v, H_Y(t(u'), v, z)) \]

\[ = (\psi \upharpoonright H^{-1}(1) (t(u'), v, z) \circ \hat{\theta}(u', K \upharpoonright H^{-1}(1) (t(u'), v, z)) + k(t(u'))) \]

\[ = (\psi \upharpoonright H^{-1}(1) (t(u'), v, z) \circ \hat{\theta}(u', K(t(u'), v, z)) \upharpoonright H^{-1}(1) + k(t(u'))) \]

On the other hand

\[ (f, H, k, K, \psi) \circ (t, T, \phi) = (f(t(u')), H(t(u')), k(t(u')), \bar{T}(u'), K(t(u'), v, z), \psi(t(u'), v, z) \circ \hat{\theta}(t(u'), v, z)) \]

applying \( R \) to this gives

\[ f(t(u'), v) \]

\[ H(t(u'), v, z) \upharpoonright H^{-1}(Y) + \bar{T}(u', K(t(u'), v, z)) \upharpoonright H^{-1}(1) \]

\[ (\psi(t(u'), v, z) \circ \hat{\theta}(u', K(t(u'), v, z))) \upharpoonright H^{-1}(1) + k(t(u')). \]

\[ \square \]
It seems clear that this proof can be carried out in the general case of cloven fibrations with the appropriate structure, but we leave that for another occasion.

One way of thinking of a map \((f, F, \phi)\) in \(\text{Dial}(A, B)\) is the following: given a witness \(u\) of \(\exists u \forall x. \alpha(u, x)\), \(f\) provides a witness \(fu\) of \(\exists v \forall y. \beta(v, y)\), and given a counter example \(y\) of \(\forall y. \beta(fu, y)\), \(F(u, y)\) is a counter example of \(\forall x. \alpha(u, x)\), and \(\phi\) is a proof of this. Now, for the Kleisli category \(\text{Dial}^+\), the difference is that given a counter example of \(\forall y. \beta(fu, y)\), \(F(u, y)\) may either give a counter example of \(\forall x. \alpha(u, x)\) or raise an exception.

In the same spirit, one may give the following intuitive characterization of the homsets \(\text{Dial}^+(A \times B, C)\) and \(\text{Dial}^+(A, B \supset C)\): The counter example part of \(\text{Dial}^+(A \times B, C)\) gives a counter example of \(\alpha\) or \(\beta\) exclusively provided a counter example of \(\gamma\). The counter example part of \(\text{Dial}^+(A, B \supset C)\) gives a counter example of \(\alpha\) or \(\beta\) or both provided a counter example of \(\gamma\). This gives some intuition as to why \(\text{Dial}^+(A, B \supset C)\) is “bigger” than \(\text{Dial}^+(A \times B, C)\).

5 Examples

Examples of fibrations that meet the conditions of Theorem 4.7 are, \(\text{cod}(\text{PER}) \to \text{PER}\) (equivalently, the split fibration \(\text{UFam}(\text{PER}) \to \text{PER}\)), \(\text{cod}(\text{Set}) \to \text{Set}\) (equivalently, the split fibration \(\text{Fam}(\text{Set}) \to \text{Set}\)), and for a topos \(\mathcal{C}\), the codomain fibration \(\text{cod}(\mathcal{C}) \to \mathcal{C}\), and the subobject fibration \(\text{Sub}(\mathcal{C}) \to \mathcal{C}\).

5.1 A modest example

We will now spell out the details of one of the examples, namely the fibration

\[
\text{UFam}(\text{PER}) \\
\downarrow \\
\text{PER}
\]

This example is important because it may provide us with the insight to give an extensional version of the Dialectica interpretation (corresponding to extensional realizability). Also some readers might like a concrete example.

**Objects** of \(\text{UFam}(\text{PER})\) are collections \((A[n])[n] \in \mathbb{N}/R\) of partial equivalence relations (pers) indexed by a per \(R\).

**Morphisms** from \((A[n])[n] \in \mathbb{N}/R\) to \((B[m])[m] \in \mathbb{N}/S\) are pairs \((u, f)\), where \(u : \mathbb{N}/R \to \mathbb{N}/S\) is a morphism in \(\text{PER}\) (that is, it is tracked by some \(e_u \in \mathbb{N}; u([n]) = [e_u \cdot n]\) and \(f = (f[n] : A[n] \to B_u([n]))\) which is tracked uniformly, i.e., there is an \(e_f \in \mathbb{N}\) such that for all \([n] \in \mathbb{N}/R\) and for all \(m \in [n]\), \(e_f \cdot m\) tracks \(f[n]([a]) = [(e_f \cdot m) \cdot a]\) for all \(m \in [n]\).

We now describe some well-known closure properties for the category \(\text{PER}\) of partial equivalence relation (also known as the category of modest sets).

The category \(\text{PER}\) has finite limits. The terminal object is given by \(\{0, 0\}\), the product of two pers \(R\) and \(S\) is given by

\[
R \times S = \{(n, m) \mid pnRp = 0\text{ and }p'nSp = 0\}.
\]
The pullback of
\[
\begin{array}{c}
S \\
\downarrow f \\
R \\
\downarrow g \\
T
\end{array}
\]
is
\[
\{ [n] \in \mathbb{N} / (R \times S) \mid g([pn]) = f([p'n]) \}
\]
The initial object in \( \text{PER} \) is the empty set. The coproduct of \( R \) and \( S \) is
\[
R + S = \{ (n, m) \mid (pn = pm = 0 \text{ and } p'nRp'm) \text{ or } (pn = pm = 1 \text{ and } p'nSp'm) \}.
\]

**Proposition 5.1.** Coproducts in \( \text{PER} \) are stable and disjoint.

Furthermore, \( \text{PER} \) has exponentials
\[
R \Rightarrow S = \{ (n, n') \mid \forall m m' R m' \Rightarrow n \cdot m S n' \cdot m' \}
\]
We also have simple products, that is, for projections \( \pi : I \times J \rightarrow I \) in \( \text{PER} \), there is a right adjoint \( \Pi_\pi \) to \( \pi^* \), it is defined as follows: For \( (R_{i,j})_{(i,j)\in I \times J} \),
\[
(\Pi_\pi R)[i] = ( \bigcap_{j \in \vert J \vert} \{ c \mid \forall n \in [j].c \cdot n \in R_{i,j}, \sim \} ), \text{ where } c \sim c' \text{ iff for all } j \in \vert J \vert. \forall n \in [j].c \cdot n R_{i,j} c' \cdot n.
\]
Now we will turn to the Dialectica-Kleisli category \( \text{Dial}^+(UFam(\text{PER})) \), for which we will describe the weak exponential in details. The product: \( (U, X, \alpha) \times (V, Y, \beta) = (U \times V, X + Y, \alpha \& \beta) \), where for \( n \in X + Y \),
\[
(\alpha \& \beta)(u, v, n) = \begin{cases} 
\alpha(u, p'n) & \text{if } pn = 0 \\
\beta(v, p'n) & \text{if } pn = 1 
\end{cases}
\]
Now \( \Box \) in the fibre over \( V \Rightarrow W \times V \times Z \Rightarrow 1 + Y \times V \times Z \) is defined by
\[
\Box(g, G, v, z) = \begin{cases} 
\beta(v, y) \Rightarrow \gamma(gv, z) & \text{if } G(v, z) = y \in Y \\
\{ (g, G, v, z) \mid G(v, z) \in 1 \} & \text{if } G(v, z) \in 1 
\end{cases}
\]
And \( \Pi_\pi \Box \) in the fibre over \( V \Rightarrow W \times V \times Z \Rightarrow 1 + Y \) is
\[
(\Pi_\pi \Box)(g, G) = ( \bigcap_{v,z} \{ k(g, G) \in \mathbb{N} \mid \forall n \in [v, z].k(g, G) \cdot n \in \Box(g, G, v, z), \sim \} ) \\
\text{where } k(g, G) \sim k'(g, G) \text{ iff } \forall v, z \in \vert V \times Z \vert \\
\forall n \in [v, z].k(g, G) \cdot n \Box(g, G, v, z) k'(g, G) \cdot n
\]
And, finally \( \beta \triangleright \gamma \) in the fibre over \( (\Pi \Box) \times V \times Z \) is
\[
(\beta \triangleright \gamma)(k(g, G) : (\Pi \Box)(g, G), v, z) = \begin{cases} 
\gamma(gv, z) & \text{if } G(v, z) \in 1 \\
\{ (k(g, G), v, z) \mid G(v, z) \in Y \} & \text{if } G(v, z) \in Y 
\end{cases}
\]
6 Conclusion and Future Work

We have shown that Dialectica categories can be generalized to cloven fibrations and how, starting with a monad, one can construct comonads on the Dialectica category. We have shown how one particular non-Girardian comonad constructed from a monad gives rise to a weakly Cartesian closed Dialectica-Kleisli category.

The ideas presented in this paper suggest two new Dialectica variants. The first one based on the new exponent as first presented in the tripos version in [BBLBCB07]. One can expand Gödel’s system T with stable, disjoint coproducts and subset types and then we can interpret implication as the new exponent. This has the advantage that we do not need the condition that primitive formulas have to be decidable (in the recursion theoretic sense). By now, this variant is described in [Bie07].

The second Dialectica variant that falls out of this work is a type theoretic one: instead of having formulas over Heyting arithmetic (this more or less corresponds to de Paiva’s original Dialectica categories) we have dependent types over some type system, and the Dialectica interpretation turns the dependent type system into a lambda calculus without the \( \eta \)-rule.

There seem to be at least two monads that give rise to comonads with interesting Kleisli categories, the free commutative monoid monad gives rise to the Diller-Nahm category and the monad \(- + 1\) gives a weakly Cartesian closed Dialectica category. One may ask if there are other comonads on Dialectica categories that gives interesting Kleisli categories. And in fact there is; this is studied in a realizability setting in [Bie08] and in a syntactical setting in [Oli08].

The PER example that we gave in Section 5.1 gives a model for an extensional version of the Dialectica interpretation, it would be interesting to describe this extensional version in details. Also the type theoretic variant of the Dialectica interpretation mentioned above deserves to be studied.

It would also be natural to find out what the closure properties are for the generalised Dialectica categories, to see if they are symmetric monoidal closed like the original ones.

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A Cartesian closure of the Cauchy completion

In this appendix we include a proof of a folklore theorem:

**Theorem A.1.** Let $C$ be a category with finite products and a weak closed structure $[B,C]$, then the Cauchy completion $\bar{C}$ is Cartesian closed.

Let $C$ be a category with finite products and a weak closed structure $[B,C]$, that is, we have a retraction

$$\mathcal{C}(A \times B, C) \xrightarrow{R} \mathcal{C}(A, [B,C])$$

onto $\mathcal{C}(A \times B, C)$ (that is, $RI = id$), natural in $A$.

In the internal language, having a weak exponent like this corresponds to having $\lambda$-calculus with the $\beta$-rule, but without the $\eta$-rule. The rule corresponding to the morphism $R$ is

$$\Gamma, x : A \vdash M : B \quad \Gamma \vdash \lambda x : A. M : A \to B \quad \Gamma \vdash N : A \to B \quad R$$

and the rule corresponding to $I$ is

$$\Gamma, x : A \vdash M : B \quad \Gamma \vdash \lambda x : A. M : A \to B \quad I$$

Consider

$$\Gamma, x : A \vdash M : B \quad \Gamma \vdash \lambda x : A. M : A \to B \quad I \quad \Gamma \vdash N : A \to B \\ R$$

Since $RI = id$ we get $M : B \overset{\beta}{=} (\lambda x : A. M)x : B$, substituting the free variable $x$ for a term $N : A$ we get the $\beta$-rule

$$(\lambda x : A. M)N : B \overset{\beta}{=} M[N/x]$$

On the other hand, consider

$$\Gamma, x : A \vdash M : B \quad \Gamma \vdash N : A \to B \quad R \quad \Gamma \vdash \lambda x : A. (N x) : A \to B \quad I$$

Since $IR \neq id$ we can conclude that

$$N : A \to B \not\overset{\eta}{=} \lambda x : A. (N x) : A \to B,$$

so we have no $\eta$-rule in the internal language. That is, not all terms of function type can be constructed by $\lambda$ abstraction.

Naturality of $I$ in $A$ is: given $a : A' \to A$, $f : A \times B \to C$,

$$I(f \circ (a \times B)) = I(f) \circ a$$

Naturality of $R$ in $A$ is: given $a : A' \to A$, $g : A \to [B,C]$,

$$R(g \circ a) = R(g) \circ (a \times B).$$
Proposition A.4. Let \( \text{ev} : [A, B] \times A \to B \) for \( R(\text{id}_{[A, B]}) \) and \( e : [A, B] \to [A, B] \) for \( I(\text{ev}) = IR(\text{id}_{[A, B]}) \).

Note that
\[
\text{ev} \circ g \times B = R(\text{id}_{B, C}) \circ (g \times B) = R(\text{id} \circ g) = R(g)
\]
and
\[
IR(g) = I(\text{ev} \circ g \times B) = I(\text{ev}) \circ g = eg
\]

It follows that
\[
\text{ev} \circ (I(f) \times B) = RI(f) = f
\]
i.e.,
\[
\tilde{f}(a)(b) = f(a, b).
\]

**Definition A.2** (Notation). For \( u : A_1 \to A \), \( x : B \to B_1 \), define \([u, x] : [A, B] \to [A_1, B_1]\)
to be
\[
I([A, B] \times A_1 \xrightarrow{[A, B] \times u} [A, B] \times A \xrightarrow{\text{ev}} B \xrightarrow{x} B_1)
\]
Observe that \([\text{id}_A, \text{id}_B] = e : [A, B] \to [A, B]\).

Proposition A.3. Let \( g : A \to [B, C] \), \( b : B' \to B \), \( c : C \to C' \). Then
\[
[b, c] \circ g = I(c \circ \text{ev} \circ [B, C] \times b) \circ g = I(c \circ \text{ev} [B, C] \times b \circ g \times B') = I(c \circ \text{ev} \circ g \times B \circ A \times b) = I(c \circ R(g) \circ A \times b).
\]

Proposition A.4. Take \( A_2 \xrightarrow{v} A_1 \xrightarrow{u} A \) and \( B \xrightarrow{x} B_1 \xrightarrow{y} B_2 \). Then
\[
[v, y] \circ [u, x] = I(y \circ R([u, x]) \circ [A, B] \times v) = I(y \circ ([A, B] \circ u \circ \text{ev} \circ x) \circ [A, B] \times v) \quad \text{as } RI = \text{id} = I((yx) \circ \text{ev} \circ ([A, B] \times uv)) = [uv, yx].
\]

We define \( \hat{C} \) to be the category of idempotents in \( C \). The objects are \((A, a)\) with \( a = a^2\) an idempotent on \( A \). The maps \((A, a) \to (B, b)\) are the maps \( f : A \to B \) with \( bfa = f \). This is called the Cauchy completion of \( C \).

\( \hat{C} \) has products. Take \((A, a), (B, b)\) in \( \hat{C} \), and consider \((A \times B, a \times b)\). For \((C, c)\) in \( \hat{C} \) consider \( f = (f_1, f_2) : C \to (A \times B) \). We see that
\[
a \times b \circ fc = f
\]
if and only if
\[
a f_1 c = f_1 \quad \text{and} \quad b f_2 c = f_2.
\]
Thus \((A \times B, a \times b)\) is the product as required.

Now take \((B, b), (C, c)\) in \( \hat{C} \). We have
\[
[b, c]^2 = [b, c] \circ [b, c] = [b^2, c^2] = [b, c].
\]

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So \([b, c]\) is an idempotent and \(([B, C], [b, c])\) is in \(\mathcal{C}\).

Suppose \(g : A \to [B, C]\) satisfies \(g = [b, c] \circ g\). Then
\[
eg g = e[b, c]g = [\text{id}_B, \text{id}_C][b, c]g = [b, c]g = g.
\]

So \(g = IR(g)\). Also
\[
R(g) = R([b, c]g) = R([b, c] \circ g \times B) = (c \circ \text{ev} \circ [B, C] \times b) \circ g \times B = cR(\text{id})(g \times B)(A \times b) = cR(g)(A \times b)
\]

That is \(R(g)\) satisfies \(R(g) = cR(g)(A \times b)\). Conversely suppose \(f : A \times B \to C\) satisfies
\[
 cf(A \times b) = f.
\]

Then
\[
[b, c] \circ I(f) = I(c \circ RI(f) \circ (A \times b)) = I(cf(A \times b)) = I(f).
\]

That is, \(I(f)\) satisfies \(I(f) = [b, c]I(f)\). It follows that \(I\) and \(R\) induce an isomorphism between maps
\[
g : (A, \text{id}_A) \to ([B, C], [b, c])
\]
and
\[
f : (A, \text{id}_A) \times (B, b) \to (C, c)
\]
in \(\mathcal{C}\). This is natural in \(A\) and the extension to an isomorphism between
\[
g : (A, a) \to ([B, C], [b, c])
\]
and
\[
f : (A, a) \times (B, b) \to (C, c)
\]
follows from: Suppose \(g = [b, c]g a\) then \(g a = [b, c]g a^2 = [b, c]g a = g\). It follows that
\[
R(g) = R(g a) = cR(g a)(A \times b) = cR(g)(a \times B)(A \times b) = cR(g)(a \times b).
\]

and conversely suppose \(f = cf(A \times b)\) then \(f(a \times B) = cf(A \times b)(a^2 \times B) = cf(A \times b)(a \times B) = f\)
and also \(f \circ (a \times B) = c(f \circ a \times B)(A \times b)\). It follows that
\[
I(f) = I(f \circ A \times b) = [b, c]I(f \circ A \times B) = [b, c]I(f)a.
\]

Thus \(([B, C][b, c])\) is the function space of \((B, b)\) to \((C, c)\) in \(\mathcal{C}\) and \(\mathcal{C}\) is Cartesian closed.
References


