The Copenhagen Interpretation

Bodil Biering

1 Introduction

We present a new functional interpretation, called the Copenhagen interpretation. Like the Diller-Nahm interpretation [DN74], the Copenhagen interpretation generalizes Gödel's Dialectica interpretation [Göd90, AF98, Sch06] from first order Heyting arithmetic $\mathbf{HA}$ to higher typed Heyting arithmetic $\mathbf{HA}^{\omega}$, or to be precise, the Dialectica interpretation requires atomic formulas to be decidable because otherwise it cannot interpret contraction, the Diller-Nahm and Copenhagen interpretations do not have this limitation. The Diller-Nahm and Copenhagen interpretations are two different ways of coping with the problem. The Diller-Nahm interpretation avoids choosing by using list types, whereas for the Copenhagen interpretation contraction is validated because the Copenhagen interpretation of conjunction is defined differently. Where the Diller-Nahm interpretation uses list types, the Copenhagen interpretation requires subset types, because the new interpretation of conjunction also forces a new interpretation of implication, and for that subset types are needed. Beside their apparent difference, the two generalizations, Diller-Nahm and Copenhagen, differs in interesting ways and this is illustrated by a detailed example.

The Copenhagen interpretation is the direct result of the categorical analysis of the Dialectica and Diller-Nahm interpretations in [dP89, Hyl02, BBLBCB07, Bie07]. The basic structure was presented in [BBLBCB07] and refined by Martin Hyland at a meeting in Copenhagen[1] in 2006, hence the name "Copenhagen interpretation". A analysis of the clause for implication can be found in [Bie07].

Apart from presenting the precise definition of the Copenhagen interpretation, there are two main results in this paper, namely the Soundness Theorem and the Axiomatization Theorem. Moreover, we state and prove precisely in what sense the Copenhagen interpretation (and in fact also the Diller-Nahm interpretation) is a generalization of Gödel's Dialectica interpretation, and finally we give a classical version of the Copenhagen interpretation.

2 Definition of Interpretation

In this section we give a complete description of the logical system that we shall use, and we also provide some models for the system. We then define the Copenhagen interpretation and illustrate the difference between the Dialectica, Diller-Nahm and Copenhagen interpretations by a detailed example.

---

[1] The authors of [BBLBCB07] have held two "Dialectica meetings" the first in Copenhagen in September 2006, the second in Genoa, June 2007

2.1 Higher Typed Heyting Arithmetic with Sum Types and Subset Types

We now describe the system $\mathbf{HA}^{\omega}_{+}$. It is essentially Gödel's system $\mathbf{T}$ (as described in [AF98] and in [Str01]) with the addition of quantifiers and sum types and subset types. For convenience we give the full definition.

Types: The set $\mathcal{T}$ of types is defined inductively as follows

- $N \in \mathcal{T}$ (type of natural numbers)
- $1 \in \mathcal{T}$ (unit or terminal type)
- $U, X \in \mathcal{T}$ then $U \times X \in \mathcal{T}$.
- $U, X \in \mathcal{T}$ then $U \Rightarrow X \in \mathcal{T}$.
- If X and U are types of $\mathcal{T}$, then $X + U$ is a type.
- If $\alpha(u,x), \beta(v,y)$ are formulas of $\mathbf{HA}^{\omega}_{+}$, then $\{(f,F) : U \Rightarrow V \times (U \times Y \Rightarrow X + 1) \mid \forall u, y.\, \mathtt{case}\, F(u,y) \in X.\alpha_C(u, F(u,y)) \to \beta_C(f(u), y)\}$ is a type.
- If s, t are closed terms of type U, then $1 + \{* : 1 \mid s =_U t\}$ is a type.

When we write

$$\mathtt{case}\, F(u,y) \in X.\alpha_C(u, F(u,y)) \to \beta_C(f(u), y)$$

it is short for

$$\left(\begin{array}{ll} \mathtt{case}\, F(u,y) \in X. & \alpha_C(u, F(u,y)) \to \beta_C(f(u), y) \\ \mathtt{case}\, F(u,y) \in 1. & \top \end{array}\right)$$

The reason for only allowing certain kinds of subset types is that we want to ensure that all types are non-empty. This rather ugly form of subset types is motivated by the fact that full subset types are not C-interpreted, this restricted form is both C-interpreted and enables us to prove the Axiomatization Theorem 5.1

Language : Countably many variables $x : U$ for each type $U \in \mathcal{T}$. The following constant symbols:

$$\begin{array}{lll} 0 : N, & \mathtt{succ} : N \to N, & \mathtt{R}^{U} : U \to (N \to U \to U) \to N \to U, \\ \mathtt{p}^{U,X} : U \to X \to U \times X, & \mathtt{pr}_0^{U,X} : U \times X \to U, & \mathtt{pr}_1^{U,X} : U \times X \to X, \\ \mathtt{inl}^{U,X} : U \to U + X, & \mathtt{inr}^{U,X} : X \to U + X, & * : 1. \end{array}$$

An equality predicate $=_U$ for each type U. For each $U, X \in \mathcal{T}$ an application operation $\mathtt{app}^{U,X}$ from $(U \to X) \times U \to X$.

Terms

$$t, s ::= \quad x : U \mid f : U \mid \mathtt{app}^{U,X}(t : U \to X, s : U) : X \mid \lambda x : U.(t : X) : U \to X$$

where f is a constant symbol, $\mathtt{app}^{U,X}(t,s)$ is often written $t(s)$. We also have term constructors $\mathtt{i}$ and $\mathtt{o}$ for subset types: if $\phi[t/x]$ is provable then $\mathtt{i}(t) : \{x : X \mid \phi\}$ is a term. If $s : \{x : X \mid \phi\}$ is a term, then $\mathtt{o}(t) : X$ is a term.

Formulas $\phi ::= s =_U t \mid \bot \mid \top \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \to \phi \mid \forall x.\phi \mid \exists x.\phi \mid \left(\begin{array}{ll} \mathtt{case}\, t \in U. & \phi \\ \mathtt{case}\, t \in X. & \phi \end{array}\right)$

By abuse of notation we will write

$$\left(\begin{array}{ll} \texttt{case}\, z \in W. & \phi(z) \\ \texttt{case}\, z \in R. & \psi(z) \end{array}\right) \quad \text{for} \quad \left(\begin{array}{ll} \texttt{case}\, z = \texttt{inl}(w). & \phi(w) \\ \texttt{case}\, z = \texttt{inr}(r). & \psi(r) \end{array}\right).$$

We notice that for every type U there is a distinguished closed term $0_U : \sigma$, defined as follows:

$$\begin{array}{l} 0_N = 0, \quad 0_{U\times X} = \texttt{p}(0_U, 0_X), \quad 0_{U\to X} = \lambda u : U.0_X, \quad 0_{U+X} = \texttt{inl}(0_U), \\ 0_{\{(f,F):U\Rightarrow V\times(U\times Y\Rightarrow X+1)\mid\forall u,y.\,\texttt{case}\,F(u,y)\in X.\alpha_C(u,F(u,y))\to\beta_C(f(u),y)\}} = \texttt{p}(\lambda u : U.0_V, \lambda u, y.\texttt{inr}(*)) \end{array}$$

The logical system for $\mathbf{HA}^{\omega}_{+}$ is:

Propositional connectives Rules have the form

$$\frac{\text{premiss}}{\text{conclusion}}$$

and

$$\begin{array}{c} P \\ \hline\hline Q \end{array} \quad \text{is short for} \quad \frac{P}{Q} \quad \text{and} \quad \frac{Q}{P}\,,$$

$$P \equiv Q \quad \text{is short for} \quad \overline{P \vdash Q} \quad \text{and} \quad \overline{Q \vdash P}$$

$$\frac{}{\phi \vdash \phi}\,(1) \qquad \frac{}{\phi \vdash \top}\,(2) \qquad \frac{}{\bot \vdash \phi}\,(3)$$

$$\frac{\phi \vdash \psi \quad \psi \vdash \theta}{\phi \vdash \theta}\,(4) \qquad \frac{}{\phi \wedge \psi \vdash \phi}\,(5) \qquad \frac{}{\phi \wedge \psi \vdash \psi}\,(6)$$

$$\frac{\phi \vdash \psi \quad \phi \vdash \chi}{\phi \vdash \psi \wedge \chi}\,(7) \qquad \frac{}{\phi \vdash \phi \vee \psi}\,(8) \qquad \frac{}{\psi \vdash \phi \vee \psi}\,(9)$$

$$\frac{\phi \vdash \chi \quad \psi \vdash \chi}{\phi \vee \psi \vdash \chi}\,(10) \qquad \begin{array}{c} \phi \vdash \psi \to \theta \\ \hline\hline \phi \wedge \psi \vdash \theta \end{array}\,(11)$$

$$(12) \quad \exists f : V^U, F : (X+1)^{U\times Y} \forall u, y. \left(\begin{array}{ll} \texttt{case}\, F(u,y) \in X. & \alpha_C(u, F(u,y)) \to \beta_C(f(u),y) \\ \texttt{case}\, F(u,y) \in 1. & \beta_C(f(u),y) \end{array}\right) \equiv$$

$$\begin{array}{l} \exists (f,F) : \{f : V^U, F : (X+1)^{U\times Y} \mid \forall u, y.\texttt{case}\, F(u,y) \in X.\alpha_C(u, F(u,y)) \to \beta_C(f(u),y)\} \\ \forall u, y. \left(\begin{array}{ll} \texttt{case}\, F(u,y) \in X. & \top \\ \texttt{case}\, F(u,y) \in 1. & \beta_C(f(u),y) \end{array}\right) \end{array}$$

$$(13) \quad \left(\begin{array}{ll} \texttt{case}\, z = \texttt{inl}(w). & \phi(w) \\ \texttt{case}\, z = \texttt{inr}(r). & \psi(r) \end{array}\right) \equiv (z = \texttt{inl}(w) \to \phi(w)) \wedge (z = \texttt{inr}(r) \to \psi(r))$$

Quantifiers

$$\begin{array}{c} \phi \vdash \psi \\ \hline\hline \phi \vdash \forall x.\psi \end{array}\; x \notin \mathrm{FV}(\phi)\,(14) \qquad \begin{array}{c} \phi \vdash \psi \\ \hline\hline \exists x.\phi \vdash \psi \end{array}\; x \notin \mathrm{FV}(\psi)\,(15)$$

Induction scheme

$$\frac{\vdash \phi(0) \quad \vdash \forall n.(\phi(n) \to \phi(\mathtt{succ}(n)))}{\vdash \forall n.\phi(n)} \quad (16)$$

Defining equations for the constants

$$\begin{array}{lll}
\neg 0 = \mathtt{succ}(x), & (\lambda x : \sigma.t)(s) = t[s/x], & \\
\mathtt{pr}_0(\mathtt{p}(x,y)) = x, & \mathtt{pr}_1(\mathtt{p}(x,y)) = y, & \mathtt{p}(\mathtt{pr}_0(z), \mathtt{pr}_1(z)) = z, \\
\mathtt{R}(x,y,0) = x, & \mathtt{R}(x,y,\mathtt{succ}(z)) = y(z, \mathtt{R}(x,y,z)), & \\
\mathtt{io}(t) = t, & \mathtt{oi}(t) = t. &
\end{array}$$

We write $z \in X$ for $\exists x : X.z = \mathtt{inl}(x)$, where $z : X + Y$.

$$\frac{}{z : X + Y \vdash \neg(z \in X \wedge z \in Y)}\ , \qquad \frac{}{z : X + Y \vdash z \in X \vee z \in Y}$$

Equality Axioms

$$\frac{}{\vdash x =_U x}\ , \quad \frac{}{x =_U y \vdash y =_U x}\ , \quad \frac{}{x =_U y \wedge y =_U z \vdash x =_U z}\ , \quad \frac{}{s = r \vdash \mathtt{app}(t,s) = \mathtt{app}(t,r)}$$

Substitution [2]

$$\frac{\phi \vdash \psi}{\phi[t/x] \vdash \psi[t/x]}\ t \text{ free for } x \text{ in } \phi, \psi \qquad \frac{t = s \vdash \psi[t/x]}{t = s \vdash \psi[s/x]}\ t, s \text{ free for } x \text{ in } \psi.$$

Models of $\mathbf{HA}_{+}^{\omega}$ Examples of models for the system $\mathbf{HA}_{+}^{\omega}$ are **HRO**, hereditary recursive operations, **HEO**, hereditary effective operations. Any subobject fibration over a category $\mathbb{C}$, where $\mathbb{C}$ is regular $(=, \exists, \wedge)$, ccc $(\times, \Rightarrow)$ and lccc $(\rightarrow, \forall)$ and has finite, stable, disjoint coproducts (needed for the `case` construction), and has a natural numbers object. Note that the Dialectica tripos defined in [BBLBCB07] is *not* a model of the system $\mathbf{HA}_{+}^{\omega}$ because it doesn't validate the rule (12).

Definition 2.1 (Copenhagen Interpretation)**.** *Suppose α and β are formulas of $\boldsymbol{HA}^{\omega}$ and $\alpha^C = \exists u \forall x.\alpha_C(u,x)$ and $\beta^C = \exists v \forall y.\beta_C(v,y)$.*

$$\begin{array}{lcl}
\alpha \in \{\top, \bot\}, & & \alpha^C = \alpha_C = \alpha. \\
(s =_U t)^C & = & \exists u : 1 + \{* : 1 \mid s =_U t\}. \begin{pmatrix} \mathtt{case} & u = \mathtt{inl}(*). & \bot \\ \mathtt{case} & u = \mathtt{inr}(*). & \top \end{pmatrix} \\
(\alpha \wedge \beta)^C & = & \exists u, v \forall z : X + Y. \begin{pmatrix} \mathtt{case} & z \in X. & \alpha_C(u,z) \\ \mathtt{case} & z \in Y. & \beta_C(v,z) \end{pmatrix} \\
(\alpha \to \beta)^C & = & \\
 & \exists \langle f, F \rangle : & \{U \Rightarrow V \times (U \times Y \Rightarrow X + 1) \mid \\
 & & \forall u, y.\mathtt{case}\, F(u,y) \in X.\alpha_C(u, F(u,y)) \to \beta_C(f(u), y)\} \\
 & \forall u, y. & \begin{pmatrix} \mathtt{case}\, F(u,y) \in 1. & \beta_C(fu, y) \\ \mathtt{case}\, F(u,y) \in X. & \top. \end{pmatrix} \\
(\forall z.\alpha(z))^C & = & \exists f : Z \to U \forall z, x.\alpha_C(z, f(z), x) \\
(\exists z.\alpha(z))^C & = & \exists z, u \forall x.\alpha_C(z, u, x) \\
(\alpha \vee \beta)^C & = & \exists z \in U + V \forall x, y. \begin{pmatrix} \mathtt{case} & z \in U. & \alpha_C(z,x) \\ \mathtt{case} & z \in V. & \beta_C(z,y) \end{pmatrix} \\
\begin{pmatrix} \mathtt{case} & z \in W. & \alpha(z) \\ \mathtt{case} & z \in R. & \beta(z) \end{pmatrix}^C & = & \exists u, v \forall x, y. \begin{pmatrix} \mathtt{case} & z \in W. & \alpha_C(z,u,x) \\ \mathtt{case} & z \in R. & \beta_C(z,v,y) \end{pmatrix}
\end{array}$$

---

[2] The second, and somewhat unusual substitution rule is provably equivalent to the standard substitution rule: $t = s \wedge \phi[t/x] \vdash \psi[s/x]$.

If we add other predicate symbols P, these are interpreted as $P^C = P_C = P$.

Note that

$$\mathsf{case}\, F(u,y) \in X.\alpha_C(u, F(u,y)) \to \beta_C(f(u), y)$$

is short for

$$\left(\begin{array}{ll} \mathsf{case}\, F(u,y) \in X. & \alpha_C(u, F(u,y)) \to \beta_C(f(u), y) \\ \mathsf{case}\, F(u,y) \in 1. & \top \end{array}\right)$$

The Copenhagen interpretation translates a formula α of $\mathbf{HA}^{\omega}_{+}$ into a formula of the form $\exists u \forall x.\alpha_C(u,x)$, where $\alpha_C(u,x)$ is a quantifier-free formula. As for the Dialectica interpretation, the most complicated case is implication. Recall that for the Dialectica interpretation we have

$$(\alpha \to \beta)^D = \exists f : U \Rightarrow V,\, F : (U \times Y) \Rightarrow X \forall u, y.\alpha_D(u, F(u,y)) \to \beta_D(f(u), y)$$

and the intuition is that given a realizer u for $\exists u \forall x.\alpha_D(u,x)$, f provides a realizer $f(u)$ for $\exists v \forall y.\beta_D(v,y)$. At the same time, if y is a counterexample of $\forall y.\beta_D(f(u),y)$ then $F(u,y)$ is a counterexample for $\forall x.\alpha_D(u,x)$.

For the C-interpretation we have a similar situation except here, F can either send a counterexample to a counterexample or $F(u,y) = * \in 1$, in which case we can think of F as raising an exception, and F can do this when we know that the conclusion, $\beta_C(fu, y)$ is true.

Definition 2.2. *A formula ϕ is said to be* C-stable *respectively* D-stable *whenever $\phi^C = \phi$, respectively $\phi^D = \phi$, where $(-)^D$ is the Dialectica interpretation and where $=$ means that they are syntactically equal.*

One important observation to make about the Copenhagen interpretation is that it is *not* the case that quantifier-free formulas are C-stable. The Dialectica interpretation enjoys the property that quantifier-free formulas are D-stable, and one can exploit that to conclude that whenever a formula has the form $\phi = \exists u \forall x.\alpha(u,x)$ where α is quantifier-free, then $\phi^D = \phi$, so in particular the D-interpretation is idempotent. For the C-interpretation we have the following for formulas A, B of the form $A = \alpha_C$ and $B = \beta_C$:

$$\begin{array}{lcl}
(A \wedge B)^C & = & \forall z : 2. \left(\begin{array}{ll} \mathsf{case}\, z = 0. & A \\ \mathsf{case}\, z = 1. & B \end{array}\right) \\
(A \to B)^C & = & \exists F : \{F : 1 \to 2 \mid \mathsf{case}\, F(*) = 0. A \to B\}. \left(\begin{array}{ll} \mathsf{case}\, F(*) = 0. & \top \\ \mathsf{case}\, F(*) = 1. & B \end{array}\right) \\
(A \vee B)^C & = & \exists z : 2. \left(\begin{array}{ll} \mathsf{case}\, z = 0. & A \\ \mathsf{case}\, z = 1. & B \end{array}\right) \\
\left(\begin{array}{ll} \mathsf{case}\, z \in X. & A(z) \\ \mathsf{case}\, z \in Y. & B(z) \end{array}\right)^C & = & \left(\begin{array}{ll} \mathsf{case}\, z \in X. & A(z), \\ \mathsf{case}\, z \in Y. & B(z) \end{array}\right) \\
(\neg A)^C & = & \exists F : \{F : 1 \to 2 \mid \mathsf{case}\, F(*) = 0. \neg A\}. \left(\begin{array}{ll} \mathsf{case}\, F(*) = 0. & \top \\ \mathsf{case}\, F(*) = 1. & \bot \end{array}\right)
\end{array}$$

Proposition 2.3. *The C-interpretation is idempotent, i.e., for all formulas ϕ, $(\phi^C)^C = \phi^C$.*

Proof: By induction on the structure of formulas, we easily show that for all α, where $\alpha^C = \exists u \forall x.\alpha_C(u,x)$, we have $\alpha_C(u,x)^C = \alpha_C(u,x)$, so

$$\begin{array}{rcl}
(\alpha^C)^C & = & (\exists u \forall x.\alpha_C(u,x))^C \\
 & = & \exists u \forall x.\alpha_C(u,x)^C \\
 & = & \exists u \forall x.\alpha_C(u,x)
\end{array}$$

□

2.2 Example

In the following we give a detailed example of a formula that have different C-, D-, and DN-interpretations.

We shall look at the formula

$$\psi = \forall f : N \to N \exists x : N(P(x) \to P(f(x)))$$

Assuming that P is a new predicate symbol over the type N, which is undecidable, we show that for the D-interpretation we are unable to define a realizer for ψ, and that the two generalizations of Dialectica, namely the Diller-Nahm and the Copenhagen interpretations give very different realizers. The point is not to try to show superiority of one interpretation over the others, but merely to illustrate the differences. Very informally, one might say that this example illustrates that while the Dialectica interpretation forces us to provide a realizer right away - whether needed or not, the Copenhagen interpretation postpones this to the very last moment. One may therefore think of the Copenhagen interpretation as a sort of call-by-name Dialectica, again this is very informal.

Classically, ψ is true, because

- If $P(f(0)) = \top$, then we put $x := 0$
- If $P(f(0)) = \bot$, then we put $x := f(0)$

The negative translation of ψ is

$$\psi^N = \forall f : N \to N \neg\neg\exists x : N(\neg\neg P(x) \to \neg\neg P(f(x)))$$

which is intuitionistically equivalent [3] to

$$\psi^N = \forall f : N \to N \neg\neg\exists x : N \neg\neg(\neg P(x) \vee P(f(x)))$$

so this holds for intuitionistic logic, $\mathbf{HA}^\omega$. Before we go on with the various functional interpretations of ψ^N, we need to consider:

Markov's principle The Dialectica interpretation validates the following version of Markov's Principle:

$$\mathrm{MP}_{\mathbf{Dia}} \qquad (\forall y \theta(y) \to \psi) \to \exists y.(\theta(y) \to \psi)$$

where θ and ψ are quantifier-free. In case θ is $\neg\phi$ and ψ is $\bot$ we have

$$\neg\forall y \neg\phi \to \exists \neg\neg\phi$$

and since intuitionistic logic validates

$$\neg\exists y \phi \leftrightarrow \forall y \neg\phi$$

we get

$$\neg\neg\exists y \phi \to \exists y \neg\neg\phi$$

---

[3] In classical logic we have $(A \to B) \leftrightarrow (\neg A \vee B)$. Suppose A and B are atomic formulas, then the negative translation of this is $(\neg\neg A \to \neg\neg B) \leftrightarrow \neg(\neg\neg A \wedge \neg B)$ and the latter is equivalent in $\mathbf{HA}$ to $\neg\neg(\neg A \vee B)$.

Thus if ϕ is double negation closed, we have

$$\mathbf{HA}^{\omega} + \mathrm{MP}_{\mathbf{Dia}} \vdash \neg\neg\exists y\phi \to \exists y\phi$$

Hence $\mathrm{MP}_{\mathbf{Dia}}$ gives us that

$$\forall f.\exists x.\neg\neg(\neg P(x) \vee P(f(x)))$$

holds in $\mathbf{HA}^{\omega} + \mathrm{MP}_{\mathbf{Dia}}$. The Dialectica interpretation of ϕ^N is thus

$$\exists F : N^N \to N \forall f : N^N.\neg\neg(\neg P(F(f)) \vee P(f(F(f))))$$

Now the term F that one would *like* to define is

$$F(f) = \begin{cases} 0 & \text{if} \quad P(f(0)) \\ f(0) & \text{if} \quad \neg P(f(0)) \end{cases}$$

But since P is not recursive, F is not either.

The Diller-Nahm interpretation[4] validates another version of MP (see [DN74]):

$$\mathrm{MP}_{\mathbf{DN}} : \qquad (\forall y A \to B) \to \exists W : \mathcal{P}_f(Y)(\forall w : W A \to B)$$

Put $B = \bot$ and $A = \neg\phi$ then we get

$$\neg\forall y\neg\phi \to \exists W : \mathcal{P}_f(Y).\neg(\forall w : W\neg\phi)$$

Since W is a finite set, $\forall w : W\neg\phi$ is a conjunction: $\bigwedge_{w\in W} \neg\phi(w)$, and in intuitionistic logic we have

$$\neg(\bigwedge_{w\in W} \neg\phi(w)) \dashv\vdash \neg\neg(\bigvee_{w\in W} \phi(w))$$

We are now able to deduce

$$\neg\neg\exists y\phi \to \exists W : \mathcal{P}_f(Y).\neg\neg(\bigvee_{w\in W} \phi(w))$$

We now turn to the Diller-Nahm interpretation of ϕ^N. Let

$$Q(f, x) = \neg P(x) \vee P(f(x)).$$

We have

$$\begin{array}{lll} \mathbf{HA}^{\omega} + \mathrm{MP}_{\mathbf{DN}} & \vdash & \forall f : N^N \exists W : \mathcal{P}_N.\neg\neg \bigvee_{w\in W} \neg\neg Q(f, w) \\ & \vdash & \forall f : N^N \exists W : \mathcal{P}_N.\neg\neg \bigvee_{w\in W} Q(f, w) \end{array}$$

The Diller-Nahm interpretation of this is

$$\exists F : (N^N) \to \mathcal{P}_f(N).\forall f : N^N.\neg\neg \bigvee_{w\in F(f)} Q(f, w).$$

We define the realizer F as

$$F(f) = \{0, f(0)\}$$

---

[4] We have not defined the type $\mathcal{P}_f(N)$ in our formal system, since it is used only for the Diller-Nahm interpretation, which is not our main focus. $\mathcal{P}_f(N)$ is the inductively defined type consisting of codes of finite sets of natural numbers by natural numbers.

and we have

$$\begin{array}{lll}\mathbf{HA}^{\omega} & \vdash & \forall f : N^N.\neg\neg(\neg P(f(0)) \vee P(f(0))) \\ & \vdash & \forall f : N^N.\neg\neg(\neg P(0) \vee P(f(0)) \vee P(f(0)) \vee Pf(f(0)))) \\ & \dashv\vdash & \forall f : N^N.\neg\neg(Q(f,0) \vee Q(f,f(0))).\end{array}$$

For the Copenhagen interpretation, things get a bit more elaborate, let us first consider the interpretation of $\neg\neg\exists u.\phi_C(u)$.

$$(\exists u.\phi_C(u) \to \bot)^C = \exists F : \{F : U \Rightarrow 2 \mid \forall u.\mathtt{case}\, Fu = 0.\neg\phi_C(u)\}\forall u : U.\gamma(F,u)$$

where

$$\gamma(F,u) = \left(\begin{array}{ll}\mathtt{case} & Fu = 1.\bot \\ \mathtt{case} & Fu = 0.\top\end{array}\right)$$

Let

$$\Box = \{F : U \Rightarrow 2 \mid \forall u.\mathtt{case}\, Fu = 0.\neg\phi_C(u)\}$$

Intuitively, $F \in \Box$ is a partial characteristic function for $\phi_C(u)$ in the sense that F might give us some information of ϕ_C, but not necessarily all. Note that, if ϕ_C is recursive, then there is a characteristic function $\chi_{\phi_C} \in \Box$.

$$\begin{array}{ll}(\neg\neg\exists u.\phi_C(u))^C = & \\ \exists G : \{G : \Box \Rightarrow U+1 \mid \forall F \in \Box.\mathtt{case}\, G(F) \in U.\neg\gamma(F,G(F)))\} & \forall F \in \Box.\alpha(G,F) = \\ \exists G : \{G : \Box \Rightarrow U+1 \mid \forall F \in \Box.\mathtt{case}\, G(F) \in U.F(GF) = 1\} & \forall F \in \Box.\alpha(G,F)\end{array}$$

where

$$\alpha(G,F) = \left(\begin{array}{ll}\mathtt{case} & G(F) \in 1.\bot \\ \mathtt{case} & G(F) \in U.\top\end{array}\right)$$

Intuitively this holds if there is a $G : \Box \Rightarrow U+1$ such that for all partial characteristic functions F, G provides a point $u \in U$ that satisfies $Fu = 1$, i.e., u is not a counter example of ϕ_C.

We are now ready to give the Copenhagen interpretation of ψ^N:

$$\begin{array}{l}(\forall f : N^N\neg\neg\exists x : N\neg\neg(\neg P(x) \vee P(f(x))))^C = \\ \exists H : N^N \to \{G : \Box \Rightarrow N+1 \mid \forall F \in \Box.\mathtt{case}\, G(F) \in N.F(G(F)) = 1\} \\ \forall f : N^N.\forall F \in \Box.\alpha(H(f),F,f)\end{array}$$

where

$$\alpha(H,F,f) = \left(\begin{array}{ll}\mathtt{case} & H(f)(F) \in 1.\bot \\ \mathtt{case} & H(f)(F) \in N.\top\end{array}\right)$$

So we to show that this holds, we must find a realizer H which for all $f : N^N$ provides a $G : \Box \Rightarrow N+1$ satisfying $\forall F \in \Box.F(G(F)) = 1$. We define H by:

$$H(f)(F) = \begin{cases} 0 & \text{if} \quad F(0) = 1 \\ f(0) & \text{if} \quad F(0) = 0 \end{cases}$$

To see that this H works we reason as follows: if $F(0) = 1$ we have found our point $G(F) \in N$ with $F(G(F)) = 1$. If $F(0) = 0$, then by definition of the type $\Box$, we have that $\neg(\neg(P(0) \vee P(f(0)))) = \neg Q(f,0)$. Now we have $\mathbf{HA}^{\omega}_{+} \vdash F(n) = 0 \vee F(n) = 1$ for all $n : N$, and by definition of $\Box$, we have $F(f(0)) = 0 \to \neg(\neg(P(f(0)) \vee P(ff(0)))) = \neg Q(f, f(0))$. Recall that we have

$$\mathbf{HA}^{\omega}_{+} \vdash \neg\neg(Q(f,0) \vee Q(f,f(0)))$$

which is equivalent to

$$\mathbf{HA}^{\omega}_{+} \vdash (\neg Q(f,0) \wedge \neg Q(f,f(0))) \to \bot$$

So assuming $F(0) = 0$ we get $\mathbf{HA}^{\omega}_{+} \vdash F(f(0)) = 0 \to \bot$ hence $\mathbf{HA}^{\omega}_{+} \vdash F(f(0)) = 1$.

3 Soundness

The soundness Theorem is the first of our two main results. It states that if a formula ϕ is provable in $\mathbf{HA}^{\omega}_{+}$ then the C-interpreted formula ϕ^C is also provable in $\mathbf{HA}^{\omega}_{+}$, which in turn means that there is a term t (which is actually a primitive recursive function in the standard model) of $\mathbf{HA}^{\omega}_{+}$ such that

$$\mathbf{HA}^{\omega}_{+} \vdash \forall x.\phi_C(t, x).$$

Lemma 3.1. *$\mathbf{HA}^{\omega}_{+} \vdash (\alpha \to \beta)^C$ iff there exists terms $g : U \Rightarrow V$, $G : (U \times Y) \Rightarrow X$ in $\mathbf{HA}^{\omega}_{+}$ such that $\mathbf{HA}^{\omega}_{+} \vdash \forall u, y.\alpha_C(u, G(u, y)) \to \beta_C(gu, y)$.*

Proof: Assume $\mathbf{HA}^{\omega}_{+} \vdash (\alpha \to \beta)^C$ this means that we have terms

$$(f, F) : \{f : V^U, F : (X+1)^{U \times Y} \mid \forall u, y.\mathtt{case}\, F(u,y) \in X.\alpha_C(u, F(u,y)) \to \beta_C(f(u), y)\}$$

such that

$$\mathbf{HA}^{\omega}_{+} \vdash \forall u, y. \begin{pmatrix} \mathtt{case}\, F(u,y) \in X. & \top \\ \mathtt{case}\, F(u,y) \in 1. & \beta_C(fu, y) \end{pmatrix}$$

Let $g = f$ and

$$G(u,y) = \begin{cases} F(u,y) & \text{if} \quad F(u,y) \in X \\ 0_X & \text{if} \quad F(u,y) \in 1 \end{cases}$$

Clearly we have

$$\mathbf{HA}^{\omega}_{+} \vdash \forall u, y.\alpha_C(u, G(u,y)) \to \beta_C(gu, y)$$

On the other hand assume we are given g, G as above, then $f = g, F = \mathtt{inl} \circ G$ have type $\{f : V^U, F : (X+1)^{U \times Y} \mid \forall u, y.\mathtt{case}\, F(u,y) \in X.\alpha_C(u, F(u,y)) \to \beta_C(f(u), y)\}$ and so we get

$$\begin{aligned} \mathbf{HA}^{\omega}_{+} \vdash \quad & \exists (f, F) : \{f : V^U, F : (X+1)^{U \times Y} \mid \forall u, y.\mathtt{case}\, F(u,y) \in X.\alpha_C(u, F(u,y)) \to \beta_C(f(u), y)\} \\ & \forall u, y. \begin{pmatrix} \mathtt{case}\, F(u,y) \in X. & \top \\ \mathtt{case}\, F(u,y) \in 1. & \beta_C(fu, y) \end{pmatrix} \end{aligned}$$

$\Box$

Theorem 3.2 (Soundness)**.** *If $\mathbf{HA}^{\omega}_{+} \vdash \alpha$ then $\mathbf{HA}^{\omega}_{+} \vdash \alpha^C$*

Proof: By induction on length of proofs in $\mathbf{HA}^{\omega}$. We show a few of the more interesting cases.

Equality We show that the transitivity rule for equality is C-interpreted, i.e., that $\mathbf{HA}^{\omega}_{+} \vdash (x = y \wedge y = z \to x = z)^C$. To do this we must find realizers

$$f : (1 + \{* : 1 \mid x = y\}) \times (1 + \{* : 1 \mid y = z\}) \to (1 + \{* : 1 \mid x = z\})$$

and

$$F : (1 + \{* : 1 \mid x = y\}) \times (1 + \{* : 1 \mid y = z\}) \Rightarrow 2$$

such that for all $u : 1 + \{* : 1 \mid x = y\}$, $v : 1 + \{* : 1 \mid y = z\}$,

$$\left(\begin{array}{ll} \mathtt{case}\, F(u,v) = 0. & \left(\begin{array}{ll} \mathtt{case}\, u = \mathtt{inl}(*). & \bot \\ \mathtt{case}\, u = \mathtt{inr}(*). & \top \end{array} \right) \\ \mathtt{case}\, F(u,v) = 1. & \left(\begin{array}{ll} \mathtt{case}\, v = \mathtt{inl}(*). & \bot \\ \mathtt{case}\, v = \mathtt{inr}(*). & \top \end{array} \right) \end{array} \right)$$

implies

$$\left(\begin{array}{ll} \mathtt{case}\, f(u,v) = \mathtt{inl}(*). & \bot \\ \mathtt{case}\, f(u,v) = \mathtt{inr}(*). & \top \end{array} \right)$$

This holds for the following definitions of f, F:

$$f(u,v) = \begin{cases} \mathtt{inr}(*) & \text{if } u = \mathtt{inr}(*) \text{ and } v = \mathtt{inr}(*) \\ \mathtt{inl}(*) & \text{otherwise} \end{cases} \qquad F(u,v) = \begin{cases} 0 & \text{if } u = \mathtt{inl}(*) \\ 1 & \text{otherwise.} \end{cases}$$

(11) We show that the rule

$$\frac{\phi \vdash \psi \to \theta}{\phi \wedge \psi \vdash \theta}\ (11)$$

is C-interpreted.

$$\mathbf{HA}^{\omega}_{+} \vdash (\phi \to (\psi \to \theta))^C \tag{1}$$

holds if and only iff

$$\begin{array}{l} \exists (g_1, g_2) : U \Rightarrow \{(f,F) : W^V \times (V \times Z)^{Y+1} \mid \forall v, z. \mathtt{case}\, F(v,z) \in Y. \psi_C(v, F(v,z) \to \theta_C(fv, z))\} \\ \exists G : U \times V \times Z \Rightarrow X \\ \forall u, v, z. \quad \phi_C(u, G(u,v,z)) \to \left(\begin{array}{ll} \mathtt{case}\, g_2(u)(v,z) \in 1. & \theta_C(g_1(u)(v), z) \\ \mathtt{case}\, g_2(u)(v,z) \in Y. & \top \end{array} \right) \end{array}$$

$$\mathbf{HA}^{\omega}_{+} \vdash (\phi \wedge \psi \to \theta)^C \tag{2}$$

holds if and only iff

$$\begin{array}{l} \exists h : U \times V \Rightarrow W, \qquad H : U \times V \times Z \Rightarrow X + Y \\ \forall u, v, z. \quad \left(\begin{array}{ll} \mathtt{case}\, H(u,v,z) \in X. & \phi_C(u, H(u,v,z)) \\ \mathtt{case}\, H(u,v,z) \in Y. & \psi_C(v, H(u,v,z)) \end{array} \right) \to \theta_C(h(u,v), z). \end{array}$$

Assume (1) holds, then we define

$$h(u,v) = g_1(u)(v), \qquad H(u,v,z) = \begin{cases} g_2(u)(v,z) & \text{if } g_2(u)(v,z) \in Y \\ G(u,v,z) & \text{otherwise.} \end{cases}$$

Assume (2) holds then we define

$$g_1(u)(v) = h(u,v), \qquad g_2(u)(v,z) = \begin{cases} H(u,v,z) & \text{if } H(u,v,z) \in Y \\ * \in 1 & \text{otherwise,} \end{cases}$$

$$G(u,v,z) = \begin{cases} H(u,v,z) & \text{if } H(u,v,z) \in X \\ 0_X & \text{otherwise.} \end{cases}$$

(14) We show that the rule number (14)

$$\frac{\phi \vdash \psi}{\phi \vdash \forall x.\psi}\; x \notin \mathrm{FV}(\phi)$$

is C-interpreted. $\mathbf{HA}^{\omega}_{+} \vdash (\phi \to \forall z.\psi(z))^C$ is equivalent to

$$\exists g : U \Rightarrow V^Z, G : Z \times U \times Y \Rightarrow X.\forall z, u, y.\phi_C(u, G(z, u, y)) \to \psi_C(z, g(u, z), y)$$

and $\mathbf{HA}^{\omega}_{+} \vdash (\phi \to \psi(z))^C$ is equivalent to

$$\exists h : Z \times U \Rightarrow V,\ H : Z \times U \times Y \Rightarrow X.\forall z, u, y.\phi_C(u, H(z, u, y)) \to \psi_C(h(z, u), y, z)$$

and these two are clearly equivalent.

(16) This is the induction scheme. Suppose that we have $u_0 : U$ such that $\forall x.\phi_C(0, u_0, x)$ and

$$\begin{array}{lll} (f, F) : & N \Rightarrow & \{(U \Rightarrow U) \times (U \times X \Rightarrow X + 1) \mid \forall u, x.\mathtt{case}\, F(n)(u, x) \in X. \\ & & \phi_C(n, u, F(n)(u, x)) \to \phi_C(\mathtt{succ}(n), f(n)(u), x)\} \\ \text{such that} & & \\ \forall n, u, x. & & \left(\begin{array}{ll} \mathtt{case}\, F(n)(u, x) \in X. & \top \\ \mathtt{case}\, F(n)(u, x) \in 1. & \phi_C(\mathtt{succ}(n), f(n)(u), x) \end{array} \right) \end{array}$$

We must provide a term $g : N \to U$ such that $\forall n, x.\phi_C(n, g(n), x)$. We define g by recursion as follows:

$$g(0) = u_0, \qquad g(\mathtt{succ}(n)) = f(n)(g(n)),$$

to be precise we use the recursion operator $\mathtt{R}$ to define g:

$$g(n) := \mathtt{R}(u_0, f, n).$$

(7) $\dfrac{\phi \vdash \psi \quad \phi \vdash \chi}{\phi \vdash \psi \wedge \chi}$ Suppose we are given $f : U \Rightarrow V, F : U \times Y \Rightarrow X$ with

$$\forall u, y.\phi_C(u, F(u, y)) \to \psi_C(fu, y)$$

and $g : U \Rightarrow W$, $G : U \times Z \Rightarrow X$ with

$$\forall u, z.\phi_C(u, G(u, z)) \to \chi_C(gv, z)$$

we must define $h : U \Rightarrow V \times W$, $H : U \times (Y + Z) \Rightarrow X$ with

$$\forall u : U, q : Y + Z.\phi_C(u, H(u, q)) \to \left(\begin{array}{ll} \mathtt{case}\, q \in Y. & \phi_C(\pi_1 h(u), q) \\ \mathtt{case}\, q \in Z. & \chi_C(\pi_2 h(u), q) \end{array} \right)$$

Clearly the following realizers satisfy the requirement:

$$h(u) = (f(u), g(u)) \qquad H(u, q) = \begin{cases} F(u, y) & \text{if} \quad q = \mathtt{inl}(y) \\ G(u, z) & \text{if} \quad q = \mathtt{inr}(z) \end{cases}$$

(13) $$\begin{pmatrix} \mathtt{case}\, z = \mathtt{inl}(w). & \phi(w) \\ \mathtt{case}\, z = \mathtt{inr}(r). & \psi(r) \end{pmatrix} \equiv (z = \mathtt{inl}(w) \to \phi(w)) \wedge (z = \mathtt{inr}(r) \to \psi(r))$$

$$\begin{pmatrix} \mathtt{case}\, z = \mathtt{inl}(w). & \phi(w) \\ \mathtt{case}\, z = \mathtt{inr}(r). & \psi(r) \end{pmatrix}^C = \exists u, v. \forall x, y. \begin{pmatrix} \mathtt{case}\, z = \mathtt{inl}(w). & \phi_C(w, u, x) \\ \mathtt{case}\, z = \mathtt{inr}(r). & \psi_C(r, v, y) \end{pmatrix}$$

and

$$\begin{array}{l} ((z = \mathtt{inl}(w) \to \phi(w)) \wedge (z = \mathtt{inr}(r) \to \psi(r)))^C = \\ \exists (u, F) : \{u : U, F : X \Rightarrow 2 \mid \forall x : X.\mathtt{case}\, Fx = 0.z = \mathtt{inl}(w) \to \phi_C(w, u, x)\} \\ \exists (v, G) : \{v : V, G : Y \Rightarrow 2 \mid \forall y : Y.\mathtt{case}\, Gy = 0.z = \mathtt{inr}(r) \to \psi_C(r, v, y)\} \\ \forall q : X + Y. \begin{pmatrix} \mathtt{case}\, q \in X. & \begin{pmatrix} \mathtt{case}\, Fx = 0. & \top \\ \mathtt{case}\, Fx = 1. & \phi_C(w, u, x) \end{pmatrix} \\ \mathtt{case}\, q \in Y. & \begin{pmatrix} \mathtt{case}\, Gy = 0. & \top \\ \mathtt{case}\, Gy = 1. & \psi_C(r, v, y) \end{pmatrix} \end{pmatrix} \end{array}$$

Showing that the former C-implies the latter, we define $h(z)(u, v) = (u, F, v, G)$ where $Fx = 1 = Gy$ for all x, y. And

$$H(z)(u, v, q) = \begin{cases} (x, 0_Y) & \text{if} \quad q = \mathtt{inl}(x) \\ (0_X, y) & \text{if} \quad q = \mathtt{inr}(y) \end{cases}$$

The other direction: $k(z)(u, F, v, G) = (u, v)$, and

$$K(z)(u, F, v, G, x, y) = \begin{cases} \mathtt{inl}(x) & \text{if} \quad z \in W \\ \mathtt{inr}(y) & \text{if} \quad z \in R. \end{cases}$$

$$\exists f : V^U, F' : (X+1)^{U \times Y} \forall u, y. \begin{pmatrix} \mathtt{case}\, F(u, y) \in X. & \alpha_C(u, F(u, y)) \to \beta_C(f(u), y) \\ \mathtt{case}\, F(u, y) \in 1. & \phi_C \end{pmatrix} \qquad \equiv$$

(12) $$\begin{array}{l} \exists (f, F) : \{f : V^U, F : (X+1)^{U \times Y} \mid \forall u, y.\mathtt{case}\, F(u, y) \in X.\alpha_C(u, F(u, y)) \to \beta_C(f(u), y)\} \\ \forall u, y. \begin{pmatrix} \mathtt{case}\, F(u, y) \in X. & \top \\ \mathtt{case}\, F(u, y) \in 1. & \phi_C \end{pmatrix} \end{array}$$

In order to show that

$$\exists f : V^U, F' : (X+1)^{U \times Y} \forall u, y. \begin{pmatrix} \mathtt{case}\, F(u, y) \in X. & \alpha_C(u, F(u, y)) \to \beta_C(f(u), y) \\ \mathtt{case}\, F(u, y) \in 1. & \phi_C \end{pmatrix} \tag{3}$$

is C-equivalent to

$$\begin{array}{l} \exists (h, H) : \{h : V^U, H : (X+1)^{U \times Y} \mid \forall u, y.\mathtt{case}\, H(u, y) \in X.\alpha_C(u, H(u, y)) \to \beta_C(h(u), y)\} \\ \forall u, y. \begin{pmatrix} \mathtt{case}\, H(u, y) \in X. & \top \\ \mathtt{case}\, H(u, y) \in 1. & \beta_C(hu, y) \end{pmatrix} \end{array} \tag{4}$$

we first show $\mathbf{HA}^{\omega}_{+} \vdash (3 \to 4)^C$ and then $\mathbf{HA}^{\omega}_{+} \vdash (4 \to 3)^C$. We start by calculating 3^C:

$$\begin{array}{lcl} 3^C & = & \exists f : V^U, F : (X+1)^{U \times Y}, G : U \times Y \Rightarrow \{z : 2 \mid \mathtt{case}\, z = 0.\alpha_C(u, F(u, y)) \to \beta_C(f(u), y)\} \\ & & \forall u, y. \begin{pmatrix} \mathtt{case}\, F(u, y) \in X. & \begin{pmatrix} \mathtt{case}\, G(u, y) = 0. & \top \\ \mathtt{case}\, G(u, y) = 1. & \beta_C(fu, y) \end{pmatrix} \\ \mathtt{case}\, F(u, y) \in 1. & \beta_C(fu, y) \end{pmatrix} \end{array}$$

$\mathbf{HA}^{\omega}_{+} \vdash (3 \to 4)^C$: We define $K(f, F, G, u, y) = u, y$ and $k(f, F, G) = (h, H)$ where $h = f$ and

$$H(u, y) = \begin{cases} F(u, y) & \text{if} \quad F(u, y) \in X \text{ and } G(u, y) = 0 \\ * \in 1 & \text{otherwise.} \end{cases}$$

$\mathbf{HA}^{\omega}_{+} \vdash (4 \to 3)^C$: Define $J(h, H, u, y) = u, y$ and $j(h, H) = (f, F, G)$ where $f = h, F = H$ and

$$G(u, y) = \begin{cases} 0 & \text{if} \quad F(u, y) \in X \\ 1 & \text{if} \quad F(u, y) \in 1. \end{cases}$$

□

4 Which Principles are Validated

In this section we show some important non-constructive principles which are *validated* by the Copenhagen interpretation (then they are said to be *C-interpreted*). In the next section we will use this to give an *axiomatization* (sometimes called a characterization) of the Copenhagen interpretation.

Definition 4.1. *A formula ϕ in the language of $\boldsymbol{HA}^{\omega}_{+}$ is said to be* validated *by the Copenhagen interpretation or* C-interpreted *if $\boldsymbol{HA}^{\omega}_{+} \vdash \phi^C$.*

Independence of Premiss

$$\text{IP} \qquad (\forall y \phi \to \exists u \forall v \psi) \to \exists u (\forall y \phi \to \forall v \psi)$$

where ϕ, ψ are of the form $\phi = \phi_C$ and $\psi = \psi_C$. First consider the premiss and conclusion of the outer $\to$:

$$\begin{aligned}
&(\forall y \phi(y) \to \exists u \forall v \psi(u, v))^C = \\
&\exists (f, F) : \Box = \{(f, F) : (1 \Rightarrow U) \times (V \Rightarrow Y + 1) \mid \forall v \in V.\mathsf{case}\, Fv \in Y.\phi(Fv) \to \psi(f(*), v)\} \\
&\forall v.\alpha(f, F, v)
\end{aligned}$$

where

$$\alpha(f, F, v) = \begin{pmatrix} \mathsf{case}\, Fv \in Y. & \top \\ \mathsf{case}\, Fv \in 1. & \psi(f(*), v) \end{pmatrix}$$

And

$$\begin{aligned}
&(\exists u (\forall y \phi \to \forall v \psi))^C = \\
&\exists u : U, G : \dagger = \{V \Rightarrow Y + 1 \mid \forall v.\mathsf{case}\, GV \in Y.\phi(Gv) \to \psi(u, v)\} \\
&\forall v.\beta(u, G, v)
\end{aligned}$$

where

$$\beta(u, G, v) = \begin{pmatrix} \mathsf{case}\, Gv \in Y. & \top \\ \mathsf{case}\, Gv \in 1. & \psi(u, v) \end{pmatrix}$$

Now we can interpret IP:

$$\begin{aligned}
&(\text{IP})^C = \\
&\exists (h, H) && : \{(h, H) : (\Box \Rightarrow U \times \dagger) \times (\Box \times V \Rightarrow V + 1) \mid \\
& && \forall (f, F), v : \Box \times V.\mathsf{case}\, H(f, F, v) \in V.\alpha((f, F), H(f, F, v)) \to \beta(h(f, F), v)\} \\
&\forall (f, F), v : \Box \times V. && \begin{pmatrix} \mathsf{case}\, H(f, F, v) \in V. & \top \\ \mathsf{case}\, H(f, F, v) \in 1. & \beta(h(f, F), v) \end{pmatrix}
\end{aligned}$$

This is realized by

$$h(f,F) = (f(*),F), \qquad H(f,F,v) = v.$$

Axiom of Choice

$$\mathrm{AC} \qquad (\forall x \exists y \phi(x,y) \to \exists F \forall x \phi(x,Fx))$$

where again we assume that ϕ is of the form $\phi = \phi_C$.

$$(\mathrm{AC})^C = (\exists F \forall x \phi(x,Fx)^C \to \exists F \forall x.\phi(x,Fx)^C)^C$$

which obviously holds.

A Generalized Markov Principle $\mathrm{MP}_{\mathbf{Dia}}$ is also validated by the C-interpretation.

$$\mathrm{MP}_C \qquad (\forall \phi(y) \to \psi) \to \exists y(\phi(y) \to \psi)$$

where ϕ, ψ are of the form $\phi = \phi_C$ and $\psi = \psi_C$. First consider

$$(\forall y \phi(y) \to \psi)^C = \exists z : \Box = \{z : Y + 1 \mid \texttt{case}\, z \in Y.\phi(z) \to \psi\}.\alpha(z)$$

where

$$\alpha(z) = \begin{pmatrix} \texttt{case}\, z \in Y. & \top \\ \texttt{case}\, z \in 1. & \psi \end{pmatrix}$$

And

$$\begin{array}{l} ((\forall y \phi(y) \to \psi) \to \exists(\phi(y) \to \psi))^C = \\ \exists (f,F) : \{(f,F) : (\Box \to Y) \times (\Box \to 2) \mid \forall z : \Box . Fz = 0.\alpha(z) \to \phi(fz) \to \psi\} \\ \forall z : \Box . \begin{pmatrix} \texttt{case}\, Fz = 0. & \top \\ \texttt{case}\, Fz = 1. & \phi(fz) \to \psi \end{pmatrix} \end{array}$$

This is realized by

$$fz = \begin{cases} z & \text{if} \quad z \in Y \\ 0 & \text{if} \quad z \in 1 \end{cases}, \qquad Fz = 0,$$

since, if $z \in 1$, we have $\alpha(z) = \psi$ and $\psi \to \phi(0) \to \psi$, and if $z \in Y$, then because $z : \Box$, we have $\phi(z) \to \psi$ and $\alpha(z) = \top$, so $\alpha(z) \to \phi(fz) \to \psi$.

5 Axiomatization

The axiomatization Theorem is our second main result. It states under certain non-constructive principles (but still in a system much more constructive than classical logic) any formula ϕ of $\mathbf{HA}^{\omega}_{+}$ is provably equivalent to it's C-interpretation ϕ^C. Moreover, ϕ is provable in the stronger system if and only if ϕ^C is provable in $\mathbf{HA}^{\omega}_{+}$. The combination of the two results of the axiomatization Theorem tells us that if a formula ϕ then we can find a realizer for the equivalent formula ϕ^C.

We have the following *axiomatization* of the functional interpretation C:

Theorem 5.1 (Axiomatization)**.** *Let* $\mathcal{X} = \{\mathrm{MP}_{\boldsymbol{Dia}}, \mathrm{AC}, \mathrm{IP}\}$*, then*

1. $\boldsymbol{HA}^{\omega}_{+} + \mathcal{X} \vdash \phi \leftrightarrow \phi^C$ *for all formulas* ϕ *of the language of* $\boldsymbol{HA}^{\omega}_{+}$.

2. $\boldsymbol{HA}^{\omega}_{+} + \mathcal{X} \vdash \phi$ *iff* $\boldsymbol{HA}^{\omega}_{+} \vdash \phi^{C}$*.*

Proof: We show 1 by induction on formulas. For $\phi \in \{\alpha \text{ atomic }, (\alpha \wedge \beta), \alpha \vee \beta, \forall z.\alpha, \exists z.\alpha\}$ it is not hard to see that $\mathbf{HA}^{\omega}_{+} \vdash \phi \leftrightarrow \phi^{C}$. For example, let us show that

$$\mathbf{HA}^{\omega}_{+} \vdash \alpha \wedge \beta \leftrightarrow (\alpha \wedge \beta)^{C}.$$

By induction we have $\mathbf{HA}^{\omega}_{+} \vdash \alpha \leftrightarrow \alpha^{C}$ and $\mathbf{HA}^{\omega}_{+} \vdash \beta \leftrightarrow \beta^{C}$, so $\mathbf{HA}^{\omega}_{+} \vdash \alpha \wedge \beta \leftrightarrow \alpha^{C} \wedge \beta^{C}$. And clearly

$$\begin{array}{lll} \mathbf{HA}^{\omega}_{+} & \vdash & \exists u, v \forall \alpha_{C}(u, x) \wedge \beta_{C}(v, y) \\ & & \leftrightarrow \exists u, v \forall z : X + Y. \left(\begin{array}{lll} \texttt{case} & z \in x. & \alpha_{C}(u, z), \\ \texttt{case} & z \in Y. & \beta_{C}(v, z) \end{array} \right) \end{array}$$

For $\phi = \left(\begin{array}{lll} \texttt{case} & z \in W. & \alpha(z), \\ \texttt{case} & z \in R. & \beta(z) \end{array} \right)^{C}$ we must use the principle IP as well:

$$\begin{array}{ll}
\left(\begin{array}{lll} \texttt{case} & z \in W. & \alpha(z), \\ \texttt{case} & z \in R. & \beta(z) \end{array} \right) & \equiv \\[2ex]
\left(\begin{array}{lll} \texttt{case} & z \in W. & \exists u \forall x.\alpha_{C}(z, u, x), \\ \texttt{case} & z \in R. & \exists v \forall y.\beta_{C}(z, v, y) \end{array} \right) & \equiv \\[2ex]
(z \in W \rightarrow \exists u \forall x.\alpha_{C}) \wedge (z \in R \rightarrow \exists v \forall y.\beta_{C}) & \equiv \text{IP} \\
\exists u(z \in W \rightarrow \forall x.\alpha_{C}) \wedge \exists v(z \in R \rightarrow \forall y.\beta_{C}) & \equiv \\
\exists u \forall x(z \in W \rightarrow \alpha_{C}) \wedge \exists v \forall y(z \in R \rightarrow \beta_{C}) & \equiv \\
\exists u, v \forall x, y.(z \in W \rightarrow \alpha_{C}) \wedge (z \in R \rightarrow \beta_{C}) & \equiv \\
\exists u, v \forall x, y. \left(\begin{array}{lll} \texttt{case} & z \in W. & \alpha_{C}(z, u, x), \\ \texttt{case} & z \in R. & \beta_{C}(z, v, y) \end{array} \right) &
\end{array}$$

And for $\phi = \alpha \rightarrow \beta$ we have the following string of equivalences:

$$\begin{array}{lll}
\exists u \forall x \alpha_{C}(u, x) \rightarrow \exists v \forall y.\beta_{C}(v, y) & \equiv & \mathbf{HA}^{\omega}_{+} \\
\forall u(\forall x \alpha_{C}(u, x) \rightarrow \exists v \forall y.\beta_{C}(v, y)) & \equiv & \text{IP} \\
\forall u \exists v(\forall x \alpha_{C}(u, x) \rightarrow \forall y.\beta_{C}(v, y)) & \equiv & \mathbf{HA}^{\omega}_{+} \\
\forall u \exists v \forall y(\forall x \alpha_{C}(u, x) \rightarrow \beta_{C}(v, y)) & \equiv & \text{MP}_{\mathbf{Dia}} \\
\forall u \exists v \forall y \exists x(\alpha_{C}(u, x) \rightarrow \beta_{C}(v, y)) & \equiv & \text{AC} \\
\exists f : V^{U}, F : X^{U \times Y} \forall u, y.\alpha_{C}(u, F(u, y)) \rightarrow \beta_{C}(f(u), y) & \equiv & \mathbf{HA}^{\omega}_{+} \\[2ex]
\exists f : V^{U}, F' : (X+1)^{U \times Y} \forall u, y. \left(\begin{array}{ll} \texttt{case}\, F(u, y) \in X. & \alpha_{C}(u, F(u, y)) \rightarrow \beta_{C}(f(u), y) \\ \texttt{case}\, F(u, y) \in 1. & \beta_{C}(fu, y) \end{array} \right) & \equiv & \mathbf{HA}^{\omega}_{+} \\[2ex]
\exists (f, F) : \{f : V^{U}, F : (X+1)^{U \times Y} \mid \forall u, y.\texttt{case}\, F(u, y) \in X.\alpha_{C}(u, F(u, y)) \rightarrow \beta_{C}(f(u), y)\} & & \\
\forall u, y. \left(\begin{array}{ll} \texttt{case}\, F(u, y) \in X. & \top \\ \texttt{case}\, F(u, y) \in 1. & \beta_{C}(fu, y) \end{array} \right) & &
\end{array}$$

Where the last two equivalences follows from 3.1.

In the previous section we showed that $\mathbf{HA}^{\omega}_{+} \vdash (\text{MP}_{\mathbf{Dia}})^{C}, (\text{IP})^{C}, (\text{AC})^{C}$, and by the Soundness Theorem 3.2, we have $\mathbf{HA}^{\omega}_{+} \vdash \phi$ implies $\mathbf{HA}^{\omega}_{+} \vdash \phi^{C}$, so it follows that

$$\mathbf{HA}^{\omega}_{+} + \mathcal{X} \vdash \phi \Rightarrow \mathbf{HA}^{\omega}_{+} \vdash \phi^{C}$$

To see that

$$\mathbf{HA}^{\omega}_{+} \vdash \phi^C \Rightarrow \mathbf{HA}^{\omega}_{+} + \mathcal{X} \vdash \phi,$$

assume $\mathbf{HA}^{\omega}_{+} \vdash \phi^C$ then also $\mathbf{HA}^{\omega}_{+} + \mathcal{X} \vdash \phi^C$, so together with 1 of Theorem 5.1 we get $\mathbf{HA}^{\omega}_{+} + \mathcal{X} \vdash \phi$. □

5.1 Copenhagen Interpretation Generalizes Dialectica Interpretation

We now show that the C-interpretation is a generalization of the D-interpretation in the sense that if require atomic formulas to be recursively decidable, then they have the same axiomatization. Moreover, ϕ^C and ϕ^D are both C- and D-equivalent, i.e., within the interpretations we can't tell the difference, assuming atomic formulas are recursively decidable.

Lemma 5.2.

$$\boldsymbol{HA}^{\omega}_{+} + \mathcal{X} \vdash \phi^D \leftrightarrow \phi^C$$

for all formulas ϕ in the language of $\boldsymbol{HA}^{\omega}_{+}$.

Proof: We have

$$\mathbf{HA}^{\omega}_{+} + \mathcal{X} \vdash \phi \leftrightarrow \phi^C$$

by Theorem 5.1. And

$$\mathbf{HA}^{\omega}_{+} + \mathcal{X} \vdash \phi \leftrightarrow \phi^D.$$

□

Theorem 5.3. *Assuming that atomic formulas are recursively decidable, the D- and C-interpretations have the same axiomatization.*

Proof: First we must expand the D-interpretation to $\mathbf{HA}^{\omega}_{+}$ by adding one clause, namely:

$$\begin{pmatrix} \texttt{case} & z \in W. & \alpha(z) \\ \texttt{case} & z \in R. & \beta(z) \end{pmatrix}^D = \exists u, v \forall x, y. \begin{pmatrix} \texttt{case} & z \in W. & \alpha_D(z, u, x) \\ \texttt{case} & z \in R. & \beta_D(z, v, y) \end{pmatrix}$$

We have (by axiomatization of Dialectica):

$$\mathbf{HA}^{\omega}_{+} + \mathcal{X} \vdash \phi \leftrightarrow \phi^D$$

and by soundness of D, under the assumption that atomic formulas are recursive,

$$\mathbf{HA}^{\omega}_{+} + \mathcal{X} \vdash \phi \Rightarrow \mathbf{HA}^{\omega}_{+} \vdash \phi^D.$$

Notice that soundness of D usually is shown only for the system $\mathbf{HA}^{\omega} + \mathcal{X}$, i.e., without the rules (12) and (13) and it only holds under the assumption that atomic formulas are recursively decidable. Rule (12) is validated by D iff atomic formulas are recursive.

□

This theorem tells us that in the system $\mathbf{HA}^{\omega}_{+}$ we might as well define the Dialectica interpretation (still under assumption that atomic formulas are recursive) as the C-interpretation, since it has the same axiomatization.

Corollary 5.4. *1.*

$$\boldsymbol{HA}^{\omega}_{+} \vdash (\phi^D \leftrightarrow \phi^C)^D$$

when atomic formulas are recursively decidable.

2.

$$\boldsymbol{HA}_{+}^{\omega} \vdash (\phi^D \leftrightarrow \phi^C)^C.$$

Proof: By Theorem 5.3

$$\mathbf{HA}_{+}^{\omega} + \mathrm{IP} + \mathrm{AC} + \mathrm{MP}_{\mathbf{Dia}} \vdash \phi^D \leftrightarrow \phi^C$$

so it follows from soundness of the two functional interpretations that

$$\mathbf{HA}_{+}^{\omega} \vdash (\phi^D \leftrightarrow \phi^C)^D \quad \text{and} \quad \mathbf{HA}_{+}^{\omega} \vdash (\phi^D \leftrightarrow \phi^C)^C$$

where we recall that soundness of D-interpretation only holds under the assumption that all atomic formulas are recursively decidable. □

6 A Classical Version of the Copenhagen Interpretation

A classical version of the C-interpretation is the double negation translation followed by the C-interpretation and then re-arranged in a way so that it becomes more simple. The classical version that we give here builds on a classical version of the D-interpretation given in [SK07]. We admit that not all of the clauses look simple, but others really are simplifications of double negation followed by C-interpretation.

We first define a negative translation $(-)'$, which was presented in [SR98] and in [SK07].

Definition 6.1. *The negative translation $A' = \neg A^*$, where A^* is defined inductively as follows:*

$$\begin{array}{lcll} P^* & = & \neg P & \textit{if } P \textit{ is prime} \\ (\neg A)^* & = & \neg A^* & \\ (A \vee B)^* & = & A^* \wedge B^* & \\ (\forall x A)^* & = & \exists x A^* & \\ (A \to B)^* & = & A' \wedge B^* & \\ (\exists x A)^* & = & \neg \exists x \neg A^* & \\ (A \wedge B)^* & = & A^* \vee B^* & \end{array}$$

Notice that the last three clauses can be defined by the others since the source is classical (**PA**). Shoenfield introduced a functional interpretation for Peano arithmetic **PA**, associating to every formula A a formula $A^S = \forall u \exists x. A_S(u,x)$, with A_S quantifier-free. Shoenfield's interpretation is defined inductively as follows:

Definition 6.2.

$$\begin{array}{lcl} P^S & = & P = P_S \quad \textit{for } P \textit{ atomic} \\ (\neg A)^S & = & \forall f \exists u. \neg A_S(u, fu) \\ (A \vee B)^S & = & \forall u, v \exists x, y A_S(u,x) \vee B_S(v,y) \\ (\forall z.A)^S & = & \forall z, u \exists x. A_S(z,u,x) \\ (A \to B)^S & = & \forall f, v \exists u, y A_S(u, f(u)) \to B_S(v,y) \\ (\exists z A)^S & = & \forall U \exists z, f. A_S(z, U(z,f), f(U(z,f))) \\ (A \wedge B)^S & = & \forall z : U + V \exists x, y. \left(\begin{array}{ll} \texttt{case}\, z \in U. & A_S(z,x) \\ \texttt{case}\, z \in V. & B_S(z,y) \end{array} \right) \end{array}$$

Let $(-)'$ be the negative translation defined in [SK07], then

Theorem 6.3. *$\boldsymbol{HA}^{\omega}_{+} + \mathcal{X} \vdash (A')^C \leftrightarrow (A^S)^C$, and $\boldsymbol{PA} \vdash A \Rightarrow \boldsymbol{HA}^{\omega}_{+} \vdash (A^S)^C$.*

Proof: From [SK07] we know that $(A')^D = \exists f \forall u A'_D(f,u)$, where $A'_D(f,u) \leftrightarrow A_S(u, f(u))$ in $\mathbf{HA}^{\omega}$. So

$$\mathbf{HA}^{\omega} \vdash (A')^D \leftrightarrow \exists f \forall u A_S(u, f(u))$$

Notice that $\exists f \forall u A_S(u, f(u)) = (A^S)^D$, so in fact

$$\mathbf{HA}^{\omega} \vdash (A')^D \leftrightarrow (A^S)^D$$

By Lemma 5.2 we have for all formulas A of $\mathbf{HA}^{\omega}_{+}$,

$$\mathbf{HA}^{\omega}_{+} + \mathcal{X} \vdash A^D \leftrightarrow A^C$$

so

$$\mathbf{HA}^{\omega}_{+} + \mathcal{X} \vdash (A')^C \leftrightarrow (A')^D \quad \text{and} \quad \mathbf{HA}^{\omega}_{+} + \mathcal{X} \vdash (A^S)^C \leftrightarrow (A^S)^D$$

it follows that

$$\mathbf{HA}^{\omega}_{+} + \mathcal{X} \vdash (A^S)^C \leftrightarrow (A')^C$$

and $(A^S)^C$ is of course C-stable. Now from [SK07] we know that [5]

$$\mathbf{PA} \vdash A \Rightarrow \mathbf{HA}^{\omega} \vdash (A^S)^D$$

hence, by the above equivalences, $\mathbf{HA}^{\omega}_{+} + \mathcal{X} \vdash (A^S)^C$ so it follows from Theorem 5.1 that

$$\mathbf{PA} \vdash A \Rightarrow \mathbf{HA}^{\omega}_{+} \vdash (A^S)^C$$

because $((A^S)^C)^C = (A^S)^C$. □

7 Acknowledgement

As mentioned in the introduction, this work relies heavily on results in [BBLBCB07] and discussions at the two "Dialectica meetings" in Copenhagen and in Genoa. The author therefore owes many, many thanks to (in alphabetic order) Lars Birkedal, Carsten Butz, Martin Hyland, Jaap van Oosten, Pino Rosolini and Thomas Streicher for encouragement, support, answering and posing of questions, endless discussions and very pleasant company. I would also like to thank Ulrich Kohlenbach for pointing out the example in Section 2.2, and Hongseok Yang for many good suggestions and discussions, and Paulo Oliva for finding a severe bug in a previous version of the paper.

A Sketch of a Simple, Higher Typed Variant of Dialectica

Just to indicate the ideas, we now give a rough sketch of a simpler variant of Dialectica, which soundly interprets $\mathbf{HA}^{\omega}$. The details will be worked out in a future paper.

In the setting of Definition 2.1, if we allow only closed atomic formulas, e.g., $s =_U t$ with s, t closed terms, then we may give the following higher typed variant of Dialectica:

---

[5] notice that to show this result one uses the fact that atomic formulas of **PA** are recursive, since atomic formulas of **PA** are equality between terms of type N.

Definition A.1. *Suppose* α *and* β *are formulas of* $\boldsymbol{HA}^\omega$ *and* $\alpha^C = \exists u \forall x. \alpha_C(u, x)$ *and* $\beta^C = \exists v \forall y. \beta_C(v, y)$.

$$\begin{array}{lll}
\alpha \in \{\top, \bot\}, & & \alpha^C = \alpha_C = \alpha. \\
\alpha \textit{ atomic, } \alpha^C & = & \exists u : 1 + \{* : 1 \mid \alpha\}. \begin{pmatrix} \texttt{case} & u = \texttt{inl}(*). & \bot \\ \texttt{case} & u = \texttt{inr}(*). & \top \end{pmatrix} \\
(\alpha \wedge \beta)^C & = & \exists u, v \forall x, y. \alpha_C(u, x) \wedge \beta_C(v, y) \\
(\alpha \to \beta)^C & = & \exists f : V^U,\ F : X^{U \times Y}. \forall u, y. \alpha_C(u, F(u, y)) \to \beta_C(fu, y) \\
(\forall z. \alpha(z))^C & = & \exists f : Z \to U \forall z, x. \alpha_C(z, f(z), x) \\
(\exists z. \alpha(z))^C & = & \exists z, u \forall x. \alpha_C(z, u, x) \\
(\alpha \vee \beta)^C & = & \exists z \in U + V \forall x, y. \begin{pmatrix} \texttt{case} & z \in U. & \alpha_C(z, x) \\ \texttt{case} & z \in V. & \beta_C(z, y) \end{pmatrix} \\
\begin{pmatrix} \texttt{case} & z \in W. & \alpha(z) \\ \texttt{case} & z \in R. & \beta(z) \end{pmatrix}^C & = & \exists u, v \forall x, y. \begin{pmatrix} \texttt{case} & z \in W. & \alpha_C(z, u, x) \\ \texttt{case} & z \in R. & \beta_C(z, v, y) \end{pmatrix}
\end{array}$$

That is, it works precisely like the original Dialectica interpretation except for the interpretation of atomic formulas, and the use of subset types. Notice that the quantifier-free part α_C is recursive by construction! This version of Dialectica works without the assumption of decidable atomic formulas, hence equality at higher types does not have to be intensional.

If we want to allow open atomic formulas $P(x)$ as well, types will become dependent, e.g., $\{x : X \mid P(x)\}$ depends on the free variable $x : X$, so if $\alpha(z)$ has a free variable $z : Z$, then

$$(\alpha(z))^C = \exists u : U(z) \forall x : X(z). \alpha_C(z, u, x)$$

and

$$(\forall z. \alpha(z))^C = \exists f : Z \to U(z). \forall z : Z, x : X(z). \alpha_C(z, f(z), x).$$

References

[AF98] Jeremy Avigad and Solomon Feferman. Gödel's functional ("Dialectica") interpretation. In *Handbook of proof theory*, volume 137 of *Stud. Logic Found. Math.*, pages 337–405. North-Holland, Amsterdam, 1998.

[BBLBCB07] J.M.E. Hyland & J. van Oosten & G. Rosolini & T. Streicher B. Biering & L. Birkedal & C. Butz. Topos theoretic versions of Dialectica interpretations. 2007. Unpublished draft.

[Bie07] Bodil Biering. Cartesian closed Dialectica categories. 2007. Submitted.

[DN74] Justus Diller and Werner Nahm. Eine Variante zur Dialectica-Interpretation der Heyting-Arithmetik endlicher Typen. *Arch. Math. Logik Grundlagenforsch.*, 16:49–66, 1974.

[dP89] V. C. V. de Paiva. The Dialectica categories. In *Categories in computer science and logic (Boulder, CO, 1987)*, volume 92 of *Contemp. Math.*, pages 47–62. Amer. Math. Soc., Providence, RI, 1989.

[Göd90] Kurt Gödel. *Collected works. Vol. II.* The Clarendon Press Oxford University Press, New York, 1990. Publications 1938–1974, Edited and with a preface by Solomon Feferman, pages 217–251.

[Hyl02] J. M. E. Hyland. Proof theory in the abstract. *Ann. Pure Appl. Logic*, 114(1-3):43–78, 2002. Commemorative Symposium Dedicated to Anne S. Troelstra (Noordwijkerhout, 1999).

[Sch06] Helmut Schwichtenberg. Proof theory. *lecture notes*, 2006.

[SK07] Thomas Streicher and Ulrich Kohlenbach. Shoenfield is Gödel after Krivine. *MLQ Math. Log. Q.*, 53(2):176–179, 2007.

[SR98] Th. Streicher and B. Reus. Classical logic, continuation semantics and abstract machines. *J. Funct. Programming*, 8(6):543–572, 1998.

[Str01] Thomas Streicher. Introduction to constructive logic and mathematics. *lecture notes*, 2000/2001.