A Variant of Gödel's Dialectica Interpretation

Bodil Biering (biering@itu.dk)

PLS Group, IT-University of Copenhagen

November 2007 / Chalmers

Brief Overview

- Recall Gödel’s Dialectica and the Diller-Nahm variant.
- The Copenhagen Interpretation.
- An illustrative example.
- Main results for the Copenhagen interpretation.
- A categorical analysis.

Recall that Gödel's Dial. translates every formula α of **HA** into a formula of the form $\alpha^D = \exists u \forall x \alpha_D(u, x)$, where α_D is a quantifier-free formula of **HA**$^\omega$ (Gödel's system **T**).

DEFINITION (DIALECTICA INTERPRETATION)

*Suppose α and β are formulas of **HA** and $\alpha^D = \exists u \forall x . \alpha_D(u, x)$ and $\beta^D = \exists v \forall y . \beta_D(v, y)$.*

$$
\begin{array}{lcl}
\alpha \text{ atomic}, & & \alpha^D = \alpha_D = \alpha. \\
(\alpha \wedge \beta)^D & = & \exists u, v \forall x, y . (\alpha_D \wedge \beta_D) \\
(\alpha \to \beta)^D & = & \exists f : U \Rightarrow V,\ F : U \times Y \Rightarrow X \forall u, y . (\alpha_D(u, F(u, y)) \to \beta_D(fu, y)) \\
(\forall z . \alpha(z))^D & = & \exists f : Z \to U \forall z, x . \alpha_D(z, f(z), x) \\
(\exists z . \alpha(z))^D & = & \exists z, u \forall x . \alpha_D(z, u, x)
\end{array}
$$

HA IS D-INTERPRETED IN HA $^\omega$

Gödel's main result was the following:

THEOREM (SOUNDNESS)

Suppose **HA** $\vdash \alpha$*, then* **HA** $^\omega \vdash \alpha^D$*, i.e., there is a term* t *of* **HA** $^\omega$ *such that* **HA** $^\omega \vdash \alpha_D(t, x)$*.*

- Soundness is shown by, for each axiom α of **HA** , finding a term (realizer) t of **HA** $^\omega$ s.t. **HA** $^\omega \vdash \alpha_D(t, y)$.
- To provide the realizer for the axiom $\alpha \to \alpha \wedge \alpha$, we need atomic formulas to be decidable (recursive).
- $$\begin{array}{l} (\alpha \to \alpha \wedge \alpha)^D = \\ \exists f : (U \times U)^U, F : X^{U \times X \times X} \\ \forall u, x, x'. \alpha_D(u, F(u, x, x')) \to \alpha_D(\pi_1 f(u), x) \wedge \alpha_D(\pi_2 f(u), x') \end{array}$$
- The terms f, F we would like to define is

$$f(u) = (u, u) \qquad F(u, x, x') = \begin{cases} x & \text{if} \quad \neg\alpha_D(u, x) \\ x' & \text{if} \quad \alpha_D(u, x) \end{cases}$$

The Diller-Nahm variant

Keep the same interpretation of conjunction, while avoiding the requirement about decidable atomic formulas.

Definition (Diller-Nahm Interpretation)

*Suppose α and β are formulas of **HA**$^\omega$ and $\alpha^{DN} = \exists u \forall x . \alpha_{DN}(u, x)$ and $\beta^{DN} = \exists v \forall y . \beta_{DN}(v, y)$.*

$$
\begin{array}{lcl}
\alpha \text{ atomic }, & & \alpha^{DN} = \alpha_{DN} = \alpha. \\
(\alpha \wedge \beta)^{DN} & = & \exists u, v \forall x, y . (\alpha_{DN} \wedge \beta_{DN}) \\
(\alpha \rightarrow \beta)^{DN} & = & \exists f : U \Rightarrow V, F : U \times Y \Rightarrow \mathbb{P}_f(X). \\
 & & \forall u, y . ((\forall x \in F(u, y) . \alpha_{DN}(u, x)) \rightarrow \beta_{DN}(fu, y)) \\
(\forall z . \alpha(z))^{DN} & = & \exists f : Z \rightarrow U \forall z, x . \alpha_{DN}(z, f(z), x) \\
(\exists z . \alpha(z))^{DN} & = & \exists z, u \forall x . \alpha_{DN}(z, u, x)
\end{array}
$$

Consider again the interpretation of $\alpha \rightarrow \alpha \wedge \alpha$:

$$
\begin{array}{l}
(\alpha \rightarrow \alpha \wedge \alpha)^{DN} = \\
\exists f : (U \times U)^U, F : \mathbb{P}_f(X)^{U \times X \times X} \\
\forall u, x, x' . (\forall z \in F(u, x, x') . \alpha_{DN}(u, z)) \rightarrow \alpha_{DN}(\pi_1 f(u), x) \wedge \alpha_{DN}(\pi_2 f(u), x')
\end{array}
$$

Now the realizer $F : U \times X \times X \rightarrow \mathbb{P}_f(X)$ is simply $F(u, x, x') = \{x, x'\}$.

The Copenhagen Interpretation

Dropping the requirement about decidability results in: the conjunction no longer satisfies $\alpha \to \alpha \wedge \alpha$. We change interpretation of conjunction:

Definition (Copenhagen Interpretation)

Suppose α and β are formulas of $\textbf{HA}^{\omega}$ *and* $\alpha^C = \exists u \forall x . \alpha_C(u, x)$ *and* $\beta^C = \exists v \forall y . \beta_C(v, y)$*, where* α_C, β_C *are atomic formulas.*

$$\alpha \text{ atomic}, \qquad \alpha^C = \alpha_C = \alpha.$$

$$(\alpha \wedge \beta)^C = \exists u, v \forall z : X + Y . \begin{pmatrix} \text{case} & z \in X . & \alpha_C(u, z), \\ \text{case} & z \in Y . & \beta_C(v, z) \end{pmatrix}$$

$$(\alpha \to \beta)^C = \exists \langle f, F \rangle : \{U \Rightarrow V \times (U \times Y \Rightarrow X + 1) \mid \forall u, y . \text{case} \quad F(u, y) \in X . \alpha_C(u, F(u, y)) \to \beta_C(f(u), y)\}$$

$$\forall u, y . \begin{pmatrix} \text{case} & F(u, y) \in 1 . & \beta_C(fu, y), \\ \text{case} & F(u, y) \in X . & \top . \end{pmatrix}$$

$$(\forall z . \alpha(z))^C = \exists f : Z \to U \forall z, x . \alpha_C(z, f(z), x)$$

$$(\exists z . \alpha(z))^C = \exists z, u \forall x . \alpha_C(z, u, x)$$

THE COPENHAGEN INTERPRETATION

Consider again the interpretation of $\alpha \to \alpha \wedge \alpha$:

$$(\alpha \to \alpha \wedge \alpha)^C =$$
$$\exists f, F : \{f : (U \times U)^U, F : (X+1)^{U \times (X+X)} \mid \forall u : U, z : X + X.\text{case} \quad F(u,z) \in X$$
$$\alpha_C(u, F(u,z)) \to \begin{pmatrix} \texttt{case} & z = \text{inl}(x). & \alpha_C(\pi_1 f(u), x), \\ \texttt{case} & z = \text{inr}(x). & \alpha_C(\pi_2 f(u), x) \end{pmatrix}\}$$

$$\forall u, z. \left(\begin{array}{lll} \texttt{case} & F(u,z) \in 1. & \begin{pmatrix} \texttt{case} & z = \text{inl}(x). & \alpha_C(\pi_1 f(u), x), \\ \texttt{case} & z = \text{inr}(x). & \alpha_C(\pi_2 f(u), x) \end{pmatrix} \\ \texttt{case} & F(u,z) \in X. & \top. \end{array} \right)$$

The realizers are $f(u) = (u, u)$, and $F : U \times (X + X) \to X$ is $F(u, \text{inl}(x)) = F(u, \text{inr}(x)) = x$.

An Example Using Equality at Higher Types

Consider the following formula of **HA** $^{\omega}$:

$$\mathbf{HA}^{\omega} \vdash \forall u, v : \mathbb{N}^{\mathbb{N}} \neg\neg\exists z : \mathbb{N}.(z = 0 \leftrightarrow u = v)$$

In the following we denote $(z = 0 \leftrightarrow u = v)$ by $Q(z, u, v)$.
The **Dialectica** interpretation of this formula is

$$\exists F : \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N} \forall u, v : \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}.\neg\neg Q(F(u, v), u, v)$$

If such an F existed it would be deciding whether $\neg\neg u = v$ or $\neg u = v$, i.e., $F(u, v) \neq 0 \leftrightarrow \neg u = v$, and $F(u, v) = 0 \leftrightarrow \neg\neg u = v$, but then we would have a realizer deciding equality between total recursive functions, which is not possible, so the formula is not D-interpreted.

An Example Using Equality at Higher Types

Using the **Diller-Nahm** interpretation we get

$$(\forall u, v : \mathbb{N}^{\mathbb{N}} \neg\neg\exists z : \mathbb{N}. Q(z, u, v))^{DN} = \exists F : \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{P}_f(\mathbb{N}). \neg\neg \bigvee_{w \in F(u,v)} Q(w, u, v)$$

so we can choose $F(u, v) = \{0, 1\}$.
Classically, it holds that

$$Q(0, u, v) \vee Q(1, u, v)$$

so intuitionistically, we have

$$\neg\neg(Q(0, u, v) \vee Q(1, u, v))$$

An Example Using Equality at Higher Types

Using the **Copenhagen** interpretation we get

$$
\begin{array}{l}
(\forall u, v : \mathbb{N}^{\mathbb{N}} \neg\neg\exists z : \mathbb{N}.Q(z, u, v))^{C} \hfill = \\
\exists H : \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to \{G : \Box \to \mathbb{N} + 1 \mid \forall F : \Box.\mathtt{case} \quad GF \in \mathbb{N}.F(GF) = 1\} \\
\forall u, v : \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \forall F : \Box.\alpha(H(u, v), F, u, v)
\end{array}
$$

where

$$\Box = \{F : \mathbb{N} \to 2 \mid \forall n : \mathbb{N}.\mathtt{case} \quad Fn = 0.\neg Q(n, u, v)\}$$

and

$$\alpha(H(u,v), F, u, v) = \begin{cases} \mathtt{case} \quad H(u,v)(F) \in 1. & \bot, \\ \mathtt{case} \quad H(u,v)(F) \in \mathbb{N}. & \top \end{cases}$$

We can choose

$$H(u,v)(F) = \begin{cases} 1 & \text{if} \quad F0 = 0 \\ 0 & \text{if} \quad F0 = 1 \end{cases}$$

this works because if $F0 = 0$ then $\neg u = v$ hence $Q(1, u, v)$ so we know that $F1 = 1$.

MAIN RESULTS FOR THE CPH INTERPRETATION

- When extending the Dialectica interpretation from first order **HA** to higher typed $\mathbf{HA}^{\omega}$, equality at higher types is forced to be intensional.
- Both Diller-Nahm and Copenhagen interpretation offers functional interpretation of $\mathbf{HA}^{\omega}$ with no restrictions on equality at higher types.
- The main results for the Copenhagen interpretation (which btw stem from the categorical analysis) are:

THEOREM (SOUNDNESS)

If $\mathbf{HA}^{\omega}_{+} \vdash \alpha$ *then* $\mathbf{HA}^{\omega}_{+} \vdash \alpha^{C}$

THEOREM (AXIOMATIZATION)

Let $\mathcal{X} = \{\mathrm{MP}_C, \mathrm{AC}, \mathrm{IP}\}$*, then*

1. $\mathbf{HA}^{\omega}_{+} + \mathcal{X} \vdash \phi \leftrightarrow \phi^{C}$ *for all formulas* ϕ *of the language of* $\mathbf{HA}^{\omega}_{+}$.
2. $\mathbf{HA}^{\omega}_{+} + \mathcal{X} \vdash \phi$ *iff* $\mathbf{HA}^{\omega}_{+} \vdash \phi^{C}$

where $\mathbf{HA}^{\omega}_{+}$ is $\mathbf{HA}^{\omega}$ with sum types and very limited version of subset types.

Categorical analysis

- Want to analyse the situation where atomic formulas are not decidable.
- Rough idea (by V. de Paiva and Hyland) is to construct category where objects correspond to α^D and a map corresponds to a realizer for $(\alpha \to \beta)^D$.
- Soundness then corresponds to showing that this category (considered as a poset) carries the structure of a Heyting algebra.

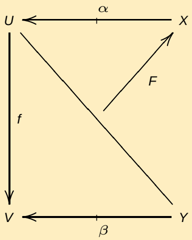
Dialectica categories

Definition

Let $\mathbb{C}$ *be a category with finite limits, then* $\mathsf{Dial}(\mathbb{C})$ *has*

Objects: *Triples* $A = (U, X, \alpha)$*, where* U, X *are obj. of* $\mathbb{C}$ *and* $\alpha \in \mathsf{Sub}(U \times X)$.

Maps: *A map from* $A = (U, X, \alpha)$ *to* $B = (V, Y, \beta)$ *is a pair of maps* (f, F) *in* $\mathbb{C}$*, such that:*



$\alpha(u, F(u, y)) \leq \beta(f(u), y)$ *in* $\mathsf{Sub}(U \times Y)$.

Theorem (VdP)

If $\mathbb{C}$ *is ccc with stable, disjoint coproducts, then* $\mathsf{Dial}(\mathbb{C})$ *has* $1, \times, \otimes, I, \multimap$.

Diller-Nahm categories

We have a comonad $! : \mathsf{Dial}(\mathbb{C}) \to \mathsf{Dial}(\mathbb{C})$ defined as follows:

$$!(U, X, \alpha) = (U, X^*, !\alpha)$$

Where X^* is the free commutative monoid, and $!\alpha$ is given by the predicate

$$!\alpha(u, e) \text{ iff } \forall x \in e.\alpha(u, x).$$

The comonad satisfies a distributive law:

$$!(A \times B) \cong !A \otimes !B$$

Definition

The Diller-Nahm category over $\mathbb{C}$ is the Kleisli category $\mathsf{Dial}_!(\mathbb{C})$ for the comonad !.

It is ccc by the following:

$$\begin{aligned} \mathsf{Dial}_!(A \times B, C) &\cong \mathsf{Dial}(!(A \times B), C) \\ &\cong \mathsf{Dial}(!A \otimes !B, C) \\ &\cong \mathsf{Dial}(!A, !B \multimap C) \\ &\cong \mathsf{Dial}_!(A, !B \multimap C) \end{aligned}$$

These are also given by a comonad, but it is not distributive so the exponential does not come automatically (as $L^+B \multimap C$).
$L^+ : \mathrm{Dial}(\mathbb{C}) \to \mathrm{Dial}(\mathbb{C})$ is given by

$$L^+(U, X, \alpha) = (U, X + 1, \hat{\alpha})$$

where

$$\hat{\alpha}(u, x) = \alpha(u, x)$$
$$\hat{\alpha}(u, *) = \top$$

DEFINITION

The Copenhagen category over $\mathbb{C}$ is the Kleisli category $\mathrm{Dial}^+(\mathbb{C})$ for the comonad L^+.

$\mathrm{Dial}^{+}(\mathbb{C})$ IS WEAKLY CARTESIAN CLOSED

- $\mathrm{Dial}^{+}(\mathbb{C})(A, B)$ is the set of realizers, $f : U \to V, F : U \times Y \to X + 1$ s.t. $\hat{\alpha}(u, F(u, y)) \leq \beta(fu, y)$.
- Under certain conditions on $\mathbb{C}$ we can define a weak exponential, i.e., there is a retract in $\mathrm{Dial}^{+}(\mathbb{C})$:

$$\mathrm{Hom}(A \times B, C) \underset{R}{\overset{I}{\rightleftarrows}} \mathrm{Hom}(A, [B, C]), \qquad RI = \mathrm{Id}$$

  natural in A.
- Using the Cauchy completion, we can obtain a Cartesian closed category.
- Naturality also ensures that the evaluation map behaves nice.

$\text{Dial}^+(\mathbb{C})$ IS WEAKLY CARTESIAN CLOSED

- For the C interpretation from **HA** into $\mathbf{HA}^{\omega}_{+}$, we only ask for existence of a realizer, this corresponds to taking the preorder reflection of $\text{Dial}^+(\mathbb{C})$ and we get an isomorphism

$$\text{Hom}(A \times B, C) \cong \text{Hom}(A, [B, C]).$$

- Our categorical results show furthermore that we can C-interpret intensional λ-calculus (β-rule, but no η-rule).

Dial^+ IS WEAKLY CARTESIAN CLOSED

We have the following theorem for subobject fibrations:

THEOREM

Let $\mathbb{C}$ be a (weakly) ccc with finite limits and stable, disjoint coproducts (i.e. $\mathbb{C}$ is extensive). Suppose we have fibred exponentials ($\rightarrow$ in $\mathrm{Sub}(X)$) and simple products ($\forall$), then $\mathrm{Dial}^+(\mathbb{C})$ is weakly Cartesian closed.

Acknowledgements

This talk builds on joint work with Lars Birkedal, Carsten Butz, Martin Hyland, Jaap van Oosten, Pino Rosolini, Thomas Streicher.