Recall from last time

Conclusion

▶ Twelf is meta logical framework.
▶ Logic Programming Semantics give raise to new function arrow.
▶ Totality = Modes + World + Termination + Coverage.

Homework

▶ Prove that if $n$ is odd and $m$ is even, then $n + m$ is odd.
▶ Extra: Implement the double negation theorem.
Today we do a larger example:
The cut-elimination for first-order logic.
Logic

Formulas \( A, B, C ::= p \mid \top \mid \bot \mid \neg A \mid A \supset B \mid \forall x. A \)

Terms \( T ::= a \)

Assumptions \( \Delta ::= \cdot \mid \Delta, A \mid \Delta; p \mid \Delta; a \)

Comments

- No predicates symbols (no equality)
- No term symbols
Sequent calculus

Judgment \( \Delta \implies A \)

Rules

- **axiom**: \( A \implies A \)
- **topR**: \( \implies \top \)
- **botL**: \( \bot \implies C \)

- **negL**: \( \Delta \implies A \) \( \frac{}{\Delta, \neg A \implies C} \)
- **negR**: \( \Delta \implies \neg A \frac{}{\Delta; p, A \implies p} \)

- **impL**: \( \Delta \implies A \) \( \Delta, B \implies C \)

- **impR**: \( \Delta, A \implies B \frac{}{\Delta \implies A \supset B} \)

- **allL**: \( \Delta, [T/x]A \implies C \)

- **allR**: \( \Delta; a \implies [a/x]A \)

\( \Delta \implies \forall x. A \)
Structural rules

\[ \Delta, A, A \Rightarrow C \quad \text{contraction} \]
\[ \frac{\Delta, A \Rightarrow C}{\Delta, A \Rightarrow C} \]

\[ \Delta \Rightarrow C \quad \text{weakening} \]
\[ \frac{\Delta, A \Rightarrow C}{\Delta, A \Rightarrow C} \]

\[ \Delta, A, B \Rightarrow C \quad \text{exchange} \]
\[ \frac{\Delta, B, A \Rightarrow C}{\Delta, B, A \Rightarrow C} \]

Observation

- Drop contraction: Affine Logic
- Drop weakening: Relevant Logic
- Drop weakening and contraction: Linear Logic
- Drop all: Lambek calculus
Sequent calculus (incl structural rules)

Judgment \[ \Delta \implies A \]

Rules

- **axiom** \( \Delta, A \implies A \)
- **topR** \( \Delta \implies \top \)
- **botL** \( \Delta, \bot \implies C \)
- **negL** \( \Delta, \neg A \implies A \)
- **negR** \( \Delta \implies \neg A \)
- **impL** \( \Delta, A \supset B \implies A \implies C \)
- **impR** \( \Delta \implies A \supset B \)
- **allL** \( \Delta, \forall x. A, [T/x]A \implies C \)
- **allR** \( \Delta \implies \forall x. A \)
Derivations

\[ D :: \Delta \Rightarrow A \Downarrow = \Delta \vdash \downarrow D \downarrow : \text{conc } A \Downarrow \]

Contexts

\[ \Gamma. \Downarrow = \cdot. \]
\[ \Gamma, A \Downarrow = \Delta \vdash A \Downarrow \]
\[ \Gamma, p \Downarrow = \Delta \vdash p : o \]
\[ \Gamma, a \Downarrow = \Delta \vdash a : i \]
World checking

**Blocks**  Defined the units of information contained in a context.

Block of *some* variables are existential quantified.

Block of *block* variables are universal.

%block l1 : some A:o block h:hyp A.
%block l2 : block a:i.
%block l3 : block p:o.

**Worlds**  Check that an operational program never puts anything else but those blocks into the context.

%worlds (l1 | l2 | l3) (hyp A) (conc A).
Theorems

Theorem In the world described by $\text{l1} \mid \text{l2} \mid \text{l3}$. If $D :: \Delta \Rightarrow A$ and $E :: \Delta, A \Rightarrow B$ then $\Delta \Rightarrow B$.

Define $\Delta \Rightarrow^* A$ all rules for $\Delta \Rightarrow A$ +

$$
\frac{\Delta \Rightarrow^* A \quad \Delta, A \Rightarrow B}{\Delta \Rightarrow^* B} \text{ cut}
$$

Theorem In the world described by $\text{l1} \mid \text{l2} \mid \text{l3}$. If $\Delta \Rightarrow^* A$ then $\Delta \Rightarrow A$.

Corollary First order logic is sound.
Proof

Schema Induction on $A$, $D$ and $E$.
Axiom conversions, essential cases, commutative cases.

Proof Blackboard

Representation Theorems.
Representation Proofs.
Conclusion

- First order logic, fits wonderfully into the framework.
- Importance of world checking.
- Proof by structural induction by cut elimination.
- Relies on coverage checking.
- Next time we will see how to conduct proofs by logical relations.