Recall from last time

Conclusion

- First order logic fits wonderfully into the framework.
- Importance of world checking.
- Proof by structural induction by cut elimination.
- Relies on coverage checking.
- Next time we will see how to conduct proofs by logical relations.
Proofs by logical relations
Going beyond inductive proofs
An application of cut-elimination.
Recall

The simply typed $\lambda$-calculus.

Types $A, B ::= o \mid A \to B$

Objects $M, N ::= x \mid M \ N \mid \lambda x. M$

Contexts $\Gamma ::= \cdot \mid \Gamma, x : A$

Remark We only consider well-typed terms.
Recall from Lecture 2

Judgments

Canonical forms \[ \Gamma \vdash M \uparrow A \]

Atomic forms \[ \Gamma \vdash N \downarrow A \]

Rules

\[ \frac{x : A \in \Gamma}{\Gamma \vdash x \downarrow A} \quad \frac{c : A \in \Sigma}{\Gamma \vdash c \downarrow A} \quad \frac{\Gamma \vdash M \downarrow A \rightarrow B \quad \Gamma \vdash N \uparrow A}{\Gamma \vdash M \ N \downarrow B} \]

\[ \frac{\Gamma \vdash M \downarrow a}{\Gamma \vdash M \uparrow a} \quad \frac{\Gamma, x : A \vdash M \uparrow B}{\Gamma \vdash \lambda x : A. M \uparrow A \rightarrow B} \]
Judgments

- Oriented definitional equality: $M \longrightarrow M'$
- Oriented definitional equality: $M \longrightarrow^* M'$
- Existence of canonical objects: $\Gamma \vdash M \uparrow M'$
- Existence of atomic objects: $\Gamma \vdash M \downarrow M'$
Conversion to Normal forms

\[
\begin{align*}
(\lambda x. M) N & \rightarrow [N/x]M \\
M & \rightarrow M' \\
\Gamma \vdash M N & \rightarrow M' N
\end{align*}
\]

\[
\begin{align*}
\Gamma \vdash x \downarrow x \\
\Gamma \vdash M \downarrow M' & \quad \Gamma \vdash N \uparrow N' \\
\Gamma \vdash M N \downarrow M' N' \\
\Gamma \vdash M \downarrow M' & \quad \Gamma \vdash M' \uparrow M'' \\
\Gamma \vdash M \uparrow M'' \\
\Gamma, x : A \vdash M x \uparrow M' \\
\Gamma \vdash M \uparrow \lambda x : A. M'
\end{align*}
\]
Goal

Theorem  For every closed $M$ of type $A$ there exists an $N$ of type $A$ s.t. $M \rightarrow^* N$ and $\vdash N \uparrow$

Question  How do we prove it?

Idea 1. Directly by induction? Doesn’t work.

Idea 2. By a logical relations argument! Yes.
The logical relation

**Definition** Logical relation

\[
\begin{align*}
\Gamma \vdash M \in \llbracket \text{o} \rrbracket & \iff \Gamma \vdash M \uparrow N \text{ for some } N \\
\Gamma \vdash M \in \llbracket A \rightarrow B \rrbracket & \iff \text{for all } \Gamma' > \Gamma \\
& \text{and for all } \Gamma' \vdash N \in \llbracket A \rrbracket \\
& \text{implies } \Gamma \vdash M \cdot N \in \llbracket B \rrbracket
\end{align*}
\]

**Observation** We use set theory as *assertion logic* to define the relation.
Meta theory

Fundamental Theorem
1. If $\Gamma \vdash M \in \llbracket A \rrbracket$ then $\Gamma \vdash M \uparrow N$ for some $N$.
2. If $\Gamma \vdash M \downarrow$ then $\Gamma \vdash M \in \llbracket A \rrbracket$.

Weak-head reduction
If $\Gamma \vdash M \in \llbracket A \rrbracket$ and $M' \rightarrow M$ then $\Gamma \vdash M' \in \llbracket A \rrbracket$.

Escape Theorem
If $\Gamma \vdash M \in \llbracket A \rrbracket$ then $\Gamma \vdash M \uparrow N$ for some $N$.

Proof see blackboard.
And into LF

Formalize assertion logic: From yesterday.
Extend logic by new judgments.

\[ \Gamma \vdash M \uparrow M' \Downarrow = \text{hc} \quad \Gamma \vdash M \Downarrow M' \Downarrow = \text{ha} \quad M \rightarrow M' \Downarrow = \text{wh} \]

Consistency of assertion logic.
Cut-elimination.
Let’s look at it in Twelf.

**Example** Combinators

\[
K : \vdash A \supset B \supset A \\
S : \vdash (A \supset B \supset C) \supset (A \supset B) \supset (A \supset C)
\]

Interpret them as \(\lambda\) terms.

\[
K = \lambda x. \lambda y. x \\
S = \lambda x. \lambda y. \lambda z. (x z) (y z)
\]

Now we show that

\[
S \ K \ K = \lambda x. x
\]
Discussion

Assertion logic

**Advantages**  Modularity
Scalability
Consistency

**Disadvantages**  Depend on Twelf’s expressive strength
Weak compared with other meta logics.
Conclusion

- Judgments and Evidence.
- Types and objects in LF type theory.
- Terms are alive, substitution is built in.
- Twelf is a good tool to experiment with ideas.
- It will talk back to you.
- Meta-theory of deductive systems.
- Many practical applications.
  - Logosphere
  - POPLmark challenge
- Proofs by structural induction.
- Proofs by logical relations.