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# Categorical Models of PILL

Rasmus Ejlers Møgelberg
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**Abstract**

We review the theory of adjunctions and comonads in the 2-category of symmetric monoidal adjunctions. This leads to the definitions of linear adjunctions, linear categories and models of DILL as in [1, 6, 7]. This theory is generalized to the fibred case, and we define models of PILL and PILL$_Y$ and morphisms between them.

## 1 Models of DILL

### 1.1 The 2-category of symmetric monoidal categories

In the following SMC stands for symmetric monoidal category.

**Definition 1.1.** A functor of SMC's from $\mathbb{C}$ to $\mathbb{C}'$ is a functor $F$ plus natural transformation

$$m\colon F(-) \otimes F(=) \Rightarrow F(- \otimes =)$$

and map $m_I\colon I \to F(I)$ satisfying the following commutative diagrams

$$\begin{array}{ccc}
(F(-) \otimes F(=)) \otimes F(\equiv) & \xrightarrow{\cong} & F(-) \otimes (F(=) \otimes F(\equiv)) \\
{\scriptstyle m\otimes id}\downarrow & & \downarrow{\scriptstyle id\otimes m} \\
F((-) \otimes (=)) \otimes F(\equiv) & & F(-) \otimes F((=) \otimes (\equiv)) \\
{\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m} \\
F(((-) \otimes (=)) \otimes (\equiv)) & \xrightarrow{\cong} & F((-) \otimes ((=) \otimes (\equiv)))
\end{array}$$

$$\begin{array}{ccc}
I \otimes F(-) & \xrightarrow{\cong} & F(-) \\
{\scriptstyle m_I\otimes id}\downarrow & & \downarrow{\scriptstyle \cong} \\
F(I) \otimes F(-) & \xrightarrow{m} & F(I \otimes (-))
\end{array}
\qquad
\begin{array}{ccc}
F(-) \otimes F(=) & \xrightarrow{\cong} & F(=) \otimes F(-) \\
{\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m} \\
F((-) \otimes (=)) & \xrightarrow{\cong} & F((=) \otimes (-))
\end{array}$$

The functor $F$ is called **strong** if the transformations $m, m_I$ are isomorphisms and **strict** if they are identities.

The composite of symmetric monoidal functors

$$(F, m^F, m_I^F)\colon \mathbb{C} \to \mathbb{C}' \text{ and } (G, m^G, m_I^G)\colon \mathbb{C}' \to \mathbb{C}''$$

is $(GF, G(m^F) \circ m^G, G(m_I^F) \circ m_I^G)$, where

$$GF(-) \otimes GF(=) \xRightarrow{m^G} G(F(-) \otimes F(=)) \xRightarrow{G(m_I^F)} GF(- \otimes =)$$
$$I \xrightarrow{m_I^G} G(I) \xrightarrow{G(m_I^F)} GF(I).$$

**Definition 1.2.** A symmetric monoidal transformation between symmetric monoidal functors $(F, m^F, m_I^F)$ and $(G, m^G, m_I^G)$ is a natural transformation $\phi \colon F \Rightarrow G$ in the usual sense satisfying

$$\begin{tikzcd} F(-) \otimes F(=) \arrow[r, "m^F"] \arrow[d, "\phi \otimes \phi"'] & F((-) \otimes (=)) \arrow[d, "\phi"] \\ G(-) \otimes G(=) \arrow[r, "m^G"] & G((-) \otimes (=)) \end{tikzcd} \qquad \begin{tikzcd} & I \arrow[dl, "m_I^F"'] \arrow[dr, "m_I^G"] & \\ F(I) \arrow[rr, "\phi_I"] & & G(I). \end{tikzcd}$$

The above defines the 2-category of SMC's. In this 2-category one can define adjunctions, monads, comonads etc. as usual. In the following we write out some of these definitions in detail.

**Definition 1.3.** A pair of functors of SMC's

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend right, "G"'] \arrow[r, phantom, "\perp"] & \mathbb{D} \arrow[l, bend right, "F"'] \end{tikzcd}$$

constitute a symmetric monoidal adjunction (with $F$ left adjoint), if $F \dashv G$ as usual and both the unit $\eta \colon id_{\mathbb{D}} \Rightarrow GF$ and the counit $\epsilon \colon FG \Rightarrow id_{\mathbb{C}}$ are symmetric monoidal transformations. This means that the following diagrams commute:

$$\begin{tikzcd} FG(-) \otimes FG(=) \arrow[r, "m^F"] \arrow[d, "\epsilon \otimes \epsilon"'] & F(G(-) \otimes G(=)) \arrow[d, "F(m^G)"] \\ (-) \otimes (=) & FG((-) \otimes (=)) \arrow[l, "\epsilon"] \end{tikzcd} \qquad \begin{tikzcd} I \arrow[d, "m_I^F"'] & \\ F(I) \arrow[r, "F(m_I^G)"'] & FG(I) \arrow[ul, "\epsilon"'] \end{tikzcd}$$

$$\begin{tikzcd} (-) \otimes (=) \arrow[r, "\eta"] \arrow[d, "\eta \otimes \eta"'] & GF((-) \otimes (=)) \\ GF(-) \otimes GF(=) \arrow[r, "m^G"] & G(F(-) \otimes F(=)) \arrow[u, "G(m^F)"'] \end{tikzcd} \qquad \begin{tikzcd} I \arrow[r, "m_I^G"] \arrow[d, "\eta"'] & G(I) \arrow[dl, "G(m_I^F)"] \\ GF(I) & \end{tikzcd}$$

The following theorem is originally from [5].

**Theorem 1.4.** *An adjunction $F \dashv G$ between symmetric monoidal categories is a symmetric monoidal adjunction iff $F$ is a strong symmetric monoidal functor.*

For a proof we refer to [8, 2]. We just note that if $(F, m^F, m_I^F)$ is strong symmetric, then the natural transformations $m^G \colon G(-) \otimes G(=) \to G((-) \otimes (=))$ is given as the adjoint correspondent to the composition

$$F(G(-) \otimes G(=)) \xrightarrow{(m^F)^{-1}} FG(-) \otimes FG(=) \xrightarrow{\epsilon \otimes \epsilon} (-) \otimes (=)$$

and the natural transformation $m_I^G$ is given as the adjoint correspondent to

$$(m_I^F)^{-1} \colon FI \to I.$$

A symmetric monoidal comonad on an SMC $\mathbb{C}$ is a vector $((F, m, m_I), \epsilon, \delta)$ such that $(F, m, m_I)$ is a SMC epifunctor, $(F, \epsilon, \delta)$ is a comonad and $\epsilon, \delta$ are symmetric monoidal transformations. Since the usual construction of a comonad from an adjunction can be carried out inside any 2-category, we obtain:

**Lemma 1.5.** *Any symmetric monoidal adjunction*

$$\mathbb{C} \ \underset{G}{\overset{F}{\leftrightarrows}}\ \mathbb{D} \qquad (\perp)$$

*gives rise to a symmetric monoidal comonad on* $\mathbb{C}$*.*

Suppose we are given a functor $F \colon \mathbb{C} \to \mathbb{D}$ between symmetric monoidal *closed* categories. Then there exists a natural transformation $n \colon F((-) \multimap (=)) \Rightarrow F(-) \multimap F(=)$ defined as

$$F((-) \multimap (=)) \longrightarrow F(-) \multimap F((-) \multimap (=)) \otimes F(-) \xrightarrow{id \multimap m}$$
$$F(-) \multimap F(((-) \multimap (=)) \otimes (-)) \xrightarrow{id \multimap F(ev)} F(-) \multimap F(=),$$

where the first map is the unit of the adjunction.

**Definition 1.6.** A morphism of SMCC's is simply a morphism of SMC's. A strong map of SMCC's is a strong map of SMC's where the transformation $n$ above is an isomorphism. The map is strict if it is a strict map of SMC's and the transformation $n$ is the identity.

## 1.2 The co-Kleisli category and the Eilenberg-Moore category of a comonad

Suppose we are given an SMC $\mathbb{C}$ and a symmetric monoidal comonad $(T, \epsilon, \delta)$ on it. We can then form the co-Kleisli category of the comonad as usual:

Objects: are the objects of $\mathbb{C}$.

Morphisms: A morphism from $X$ to $Y$ is a morphism in $\mathbb{C}$ from $TX$ to $Y$.

Composition: Composition of maps $f \colon X \to Y$ and $g \colon Y \to Z$ is given as

$$TX \xrightarrow{\delta_X} T^2X \xrightarrow{Tf} TY \xrightarrow{g} Z.$$

The natural transformation $\epsilon$ plays the role of the identity.

We denote the co-Kleisli category by $\mathbb{C}_T$.

We can also form the Eilenberg-Moore category of the comonad as

Objects: Coalgebras for the comonad, i.e., maps $h \colon X \to TX$ satisfying

$$\begin{array}{ccc} X & \xrightarrow{h} & TX \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \delta} \\ TX & \xrightarrow{Th} & T^2X \end{array} \qquad\qquad \begin{array}{ccc} X & \xrightarrow{h} & TX \\ & {\scriptstyle id}\searrow & \downarrow{\scriptstyle \epsilon} \\ & & X \end{array}$$

Morphisms: Morphisms of coalgebras

We denote the Eilenberg-Moore category by $\mathbb{C}^T$.

**Lemma 1.7.** *The co-Kleisli category of a comonad is isomorphic to the full subcategory of the Eilenberg-Moore category on the free coalgebras for the comonad, i.e., the coalgebras of the form* $\delta_X\colon T(X) \to T^2(X)$.

*Proof.* There is clearly a bijective correspondence between objects. We need to check that this correspondence extends to morphisms. Suppose $h\colon TX \to Y$ is a morphism in the co-Kleisli category from $X$ to $Y$. Then $Th \circ \delta_X$ defines a morphism of coalgebras since

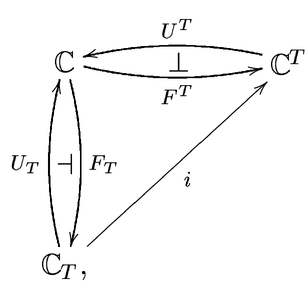
$$\begin{array}{ccccc} T^2X & \xrightarrow{T\delta} & T^3X & \xrightarrow{T^2h} & T^2Y \\ {\scriptstyle \delta_X}\uparrow & & {\scriptstyle \delta_{TX}}\uparrow & & \uparrow{\scriptstyle \delta_Y} \\ TX & \xrightarrow{\delta_X} & T^2X & \xrightarrow{Th} & TY, \end{array}$$

where the square to the left commutes by the definition of comonad, and the diagram to the right commutes by naturality of $\delta$. To check that this defines a functor from the co-Kleisli category to the Eilenberg-Moore category, suppose $h\colon TX \to Y$ and $h'\colon TY \to Z$ in $\mathbb{C}$. If we first use the functor and then compose, we obtain $(Th') \circ \delta_Y \circ (Th) \circ \delta_X$. If we first compose and then apply the functor, we obtain $T(h' \circ Th \circ \delta_X) \circ \delta_X$, which by definition of comonad is $Th' \circ T^2h \circ \delta_{TX} \circ \delta_X$. By naturality of $\delta$, we conclude that the functor commutes with composition. Clearly $\epsilon$ is mapped to the identity.

Suppose on the other hand that $f\colon TX \to TY$ defines a map of coalgebras, i.e., $Tf \circ \delta_X = \delta_Y \circ f$. Then we can define the map $\epsilon_Y \circ f\colon TX \to Y$, which is a map in the co-Kleisli category from $X$ to $Y$. Again we need to check that this defines a functor. Suppose $f'\colon TY \to TZ$ is another map of coalgebras. Composing first and the applying the functor gives $\epsilon_Z \circ f' \circ f$. Applying the functor first and then composing gives $\epsilon_Z \circ f' \circ T(\epsilon_Y \circ f) \circ \delta_X = \epsilon_Z \circ f' \circ T(\epsilon_Y) \circ \delta_Y \circ f$, since $f$ is a map of coalgebras. We now use $T(\epsilon_Y) \circ \delta_Y = id_Y$ by one of the equations for comonads to conclude that the functor commutes with composition. Clearly the identity is mapped to $\epsilon$.

We need to check that the two functors are inverses of each other. Suppose we start with a map in the co-Kleisli category, i.e., a map $h\colon TX \to Y$. Applying the two functors to this gives $\epsilon_Y \circ Th \circ \delta_X = h \circ \epsilon_{TX} \circ \delta_X = h$. If we start with a map of coalgebras $f\colon TX \to TY$, applying the two functors gives $T(\epsilon_Y) \circ T(f) \circ \delta_X = T(\epsilon_Y) \circ \delta_Y \circ f = f$. ◻

We have the usual adjunctions between $\mathbb{C}$ and $\mathbb{C}^T$ and $\mathbb{C}$ and $\mathbb{C}_T$. We can illustrate these as




where $i$ is the inclusion. We know that $iF_T = F^T, U^T i = U_T$. Without further assumptions, neither $\mathbb{C}_T$ nor $\mathbb{C}^T$ have a natural SMC structure, so it does not make sense to ask for the adjunctions to be symmetric monoidal.

**Definition 1.8.** A *linear adjunction* is a symmetric monoidal adjunction

$$\mathbb{C} \underset{G}{\overset{F}{\leftrightarrows}} \mathbb{D}, \quad F \dashv G$$

where the SMC-structure on $\mathbb{D}$ is in fact a cartesian structure, and $\mathbb{C}$ is SMCC.

A morphism of linear adjunctions from $\mathbb{C} \underset{G}{\overset{F}{\leftrightarrows}} \mathbb{D}$ to $\mathbb{C}' \underset{G'}{\overset{F'}{\leftrightarrows}} \mathbb{D}'$ is a pair of functors $H, K$ where $H$ is a strict map of symmetric monoidal closed categories, and $K$ is a strong map of symmetric monoidal categories such that the diagrams

$$\begin{array}{ccccc} \mathbb{C} & \xrightarrow{G} & \mathbb{D} & \xrightarrow{F} & \mathbb{C} \\ \downarrow{\scriptstyle H} & & \downarrow{\scriptstyle K} & & \downarrow{\scriptstyle H} \\ \mathbb{C}' & \xrightarrow{F'} & \mathbb{D}' & \xrightarrow{G'} & \mathbb{C}' \end{array}$$

commute *up to isomorphism*. Furthermore, $H$ is required to commute with the comonads induced by the adjunctions, i.e., $HFG = F'G'H$, $H\epsilon = \epsilon' H$ and $H\delta = \delta' H$, where $\delta, \delta'$ are the comultiplications of the comonad induced by the adjunctions.

A natural transformation from $(H, K)$ to $(H, K')$ (notice that the first components of the two functors are equal) is a natural transformation from $K$ to $K'$.

The definition of natural transformation may seem a bit unintuitive, in particular the fact that natural transformations are always identity on the SMCC components of a functor. We have chosen this definition because we want a fairly restrictive notion of equivalence between linear adjunctions.

It is well-known that DILL can be interpreted soundly and completely in any linear adjunction [1]

**Remark 1.9.** An LNL-model is a linear adjunction in which the cartesian category is closed.

**Definition 1.10.** A linear category is an SMCC $\mathbb{C}$ with a symmetric monoidal comonad $((!, m, m_I), \epsilon, \delta)$ and symmetric monoidal natural transformations $e\colon !(-) \to I$, $d\colon !(-) \to !(-) \otimes !(-)$, such that

- For each object $A$, $(!A, e_A, d_A)$ is a commutative comonoid, i.e.,

$$\begin{array}{ccc} !A & \xrightarrow{d_A} & !A \otimes !A \\ & {\scriptstyle \cong} \searrow & \downarrow{\scriptstyle id \otimes e_A} \\ & & !A \otimes I \end{array} \qquad \begin{array}{ccc} !A & \xrightarrow{d_A} & !A \otimes !A \\ & {\scriptstyle d_A} \searrow & \downarrow{\scriptstyle s} \\ & & !A \otimes !A \end{array}$$

$$\begin{array}{ccccc} !A & \xrightarrow{d_A} & !A \otimes !A & \xrightarrow{id \otimes d_A} & !A \otimes (!A \otimes !A) \\ \downarrow{\scriptstyle d_A} & & & & \downarrow{\scriptstyle \cong} \\ !A \otimes !A & & \xrightarrow{d_A \otimes id} & & (!A \otimes !A) \otimes !A, \end{array}$$

  where $s$ is the natural transformation $(-) \otimes (=) \Rightarrow (=) \otimes (-)$,

- $e_A, d_A$ define coalgebra maps from $\delta_A\colon !A \to !!A$ to the coalgebras $m_I\colon I \to !I$ and

$$!A \otimes !A \xrightarrow{\delta_A \otimes \delta_A} !!A \otimes !!A \xrightarrow{m} !(!A \otimes !A)$$

- All coalgebra maps between free coalgebras preserve the comonoid structure, i.e., if $f\colon !A \to !B$ is such that

$$\begin{array}{ccc} !!A & \xrightarrow{!f} & !!B \\ {\scriptstyle\delta_A}\uparrow & & \uparrow{\scriptstyle\delta_B} \\ !A & \xrightarrow{f} & !B \end{array}$$

then

$$\begin{array}{ccc} !A & \xrightarrow{f} & !B \\ & {\scriptstyle e_A}\searrow \quad \swarrow{\scriptstyle e_B} & \\ & I & \end{array} \qquad \begin{array}{ccc} !A & \xrightarrow{f} & !B \\ {\scriptstyle d_A}\downarrow & & \downarrow{\scriptstyle d_B} \\ !A\otimes !A & \xrightarrow{f\otimes f} & !B\otimes !B. \end{array}$$

Linear categories model Intuitionistic Linear Logic (ILL). In ILL, types of the form $!A$ behave intuitionistically, and intuitively, one should think of $e$ as providing weakening for these types, and $d$ as providing contraction.

**Lemma 1.11.** *In Definition 1.10, the last condition can be replaced by the condition that $\delta$ preserves comonoid structure.*

*Proof.* From the definition of comonads, we see that $\delta$ is a coalgebra map, and thus the new condition is a special case of the old.

For the other implication, suppose that $f\colon !A \to !B$ is a map of coalgebras. Then

$$e_B \circ f = e_{!B} \circ \delta_B \circ f = e_{!B} \circ (!f) \circ \delta_A = e_{!A} \circ \delta_A = e_A$$

which proves commutativity of the first diagram. For the second notice first that

$$f = !\epsilon_B \circ \delta_B \circ f = !\epsilon_B \circ (!f) \circ \delta_A.$$

The result now follows from the following commutative diagram:

$$\begin{array}{ccccccc} !A & \xrightarrow{\delta} & !!A & \xrightarrow{!f} & !!B & \xrightarrow{!\epsilon} & !B \\ {\scriptstyle d_A}\downarrow & & \downarrow{\scriptstyle d_{!A}} & & \downarrow{\scriptstyle d_{!B}} & & \downarrow{\scriptstyle d_B} \\ !A\otimes !A & \xrightarrow{\delta\otimes\delta} & !!A\otimes !!A & \xrightarrow{!f\otimes !f} & !!B\otimes !!B & \xrightarrow{!\epsilon\otimes !\epsilon} & !B\otimes !B. \end{array}$$

□

**Definition 1.12.** A morphism of linear categories from $(\mathbb{C}, !, d, e)$ to $(\mathbb{C}', !', d', e')$ is a strong symmetric monoidal closed functor $F$ preserving all the comonad structure on the nose, i.e., $!'F = F!, \epsilon' F = F\epsilon, \delta' F = F\delta$. If the functor $F$ is strict, we call this a strict functor of linear categories.

**Lemma 1.13.** *For a linear category, the associated Eilenberg-Moore category is cartesian.*

*Proof.* The product of two coalgebras $h_A\colon A \to !A, h_B\colon B \to !B$ is

$$A \otimes B \xrightarrow{h_A \otimes h_B} !A \otimes !B \xrightarrow{m} !(A \otimes B)$$

with projection given by

$$A \otimes B \xrightarrow{id\otimes h_B} A \otimes !B \xrightarrow{id\otimes e_B} A \otimes I \xrightarrow{\cong} A$$

and diagonal $\Delta_A$ given by

$$A \xrightarrow{h_A} !A \xrightarrow{d_A} !A \otimes !A \xrightarrow{\epsilon_A \otimes \epsilon_A} A \otimes A.$$

Having defined the diagonal, pairing of functions $f\colon A \to B, g\colon A \to C$ is defined as usual as $\langle f, g\rangle = f \otimes g \circ \Delta_A$.

The terminal object is $m_I\colon I \to !I$. □

**Proposition 1.14.** *Each linear adjunction*

$$\mathbb{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathbb{D} \quad (F \dashv G)$$

*gives rise to a linear category whose comonad is* $! = FG$*. This extends to a functor from the category of linear adjunctions to the category of linear categories with strict morphisms.*

*Proof.* Recall first that in a linear adjunction, the left adjoint is strong by Theorem 1.4, i.e., $m, m_I$ are isomorphisms.

The map $e_A$ is the composition

$$FGA \xrightarrow{F(\star)} F(1) \xrightarrow{m_I^{-1}} I$$

and $d_A$ is

$$FGA \xrightarrow{F(\Delta)} F(GA \times GA) \xrightarrow{m^{-1}} FGA \otimes FGA.$$

For the details of this proof, we refer to [2].

The last part of the proposition is obvious. □

## 1.3 The category of products of free coalgebras

Given a linear category $(\mathbb{C}, !, e, d)$ we define $\mathbb{C}_!^\star$ to have as objects finite vectors of objects of $\mathbb{C}$ and as morphisms from $(A_i)$ to $(B_j)$ morphisms of $\mathbb{C}^!$ from $\prod \delta_{A_i}$ to $\prod \delta_{B_j}$. This category is equivalent to the full subcategory of $\mathbb{C}^!$ on products of objects of $\mathbb{C}_!$. We call $\mathbb{C}_!^\star$ the category of products of free coalgebras and we will often denote an object of $\mathbb{C}_!^\star$ simply as $\prod \delta_{A_i}$ instead of $(A_i)$.

**Lemma 1.15.** *Given a linear category* $(\mathbb{C}, !, e, d)$*, there is a symmetric monoidal adjunction*

$$\mathbb{C} \underset{F_!^\star}{\overset{U_!^\star}{\rightleftarrows}} \mathbb{C}_!^\star \quad (F_!^\star \dashv U_!^\star),$$

*i.e., a linear adjunction whose associated linear category (Proposition 1.14) is isomorphic to* $(\mathbb{C}, !, e, d)$*.*

*Proof.* This is basically the restriction of the adjunction between $\mathbb{C}^!$ and $\mathbb{C}$. To show that the adjunction is symmetric monoidal, it suffices to show that $U_!^\star$ is a strong symmetric monoidal functor. But

$$U_!^\star((A_i) \times (B_j)) = U_!^\star((A_1, \ldots A_n, B_1 \ldots B_m) = !A_1 \otimes \ldots !A_n \otimes !B_1 \otimes \ldots B_m$$

and

$$U_!^\star(A_i) \otimes U_!^\star(B_j) = (\otimes_i !A_i) \otimes (\otimes_j !B_j)$$

so $U_!^\star$ is clearly a strong symmetric monoidal functor. □

**Lemma 1.16.** *The construction of Lemma 1.15 extends to a functor from the category of linear categories with strict maps to the category of linear adjunctions. This functor is right inverse to the functor of Proposition 1.14.*

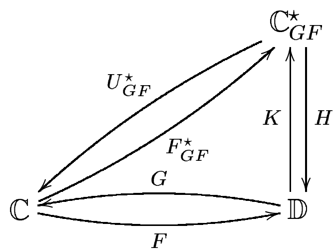
*Proof.* Suppose $K\colon (\mathbb{C}, !) \to (\mathbb{D}!')$ is a map of linear categories. We define $H\colon \mathbb{C}^\star_! \to \mathbb{D}^\star_{!'}$ by $H(A_i) = (KA_i)$ and on morphisms

$$H(h\colon \otimes !A_i \to \otimes !B_j) = K(h)\colon \otimes !KA_i = K(\otimes !A_i) \to K(\otimes !B_j) = \otimes !KB_j.$$

The reader may verify that because $K$ is strict and commutes with $\delta$, this defines a map of coalgebras. Clearly $H$ is a strict map of SMC's and the two required diagrams commute on the nose. ☐

**Definition 1.17.** We define the category of DILL models to be the full subcategory of the category of linear adjunctions on the objects equivalent to the objects induced by linear categories as in Lemma 1.15

If we write out the definition above, then a DILL model is a linear adjunction $\mathbb{C} \underset{F}{\overset{G}{\underset{\perp}{\rightleftarrows}}} \mathbb{D}$ such that there exists maps of SMC's $H, K$ as in




such that $H, K$ is an equivalence of categories and such that

$$G \cong U^\star_{GF}K, \quad F^\star_{GF} \cong KF, \quad GH \cong U^\star_{GF}, \quad HF^\star_{GF} \cong F.$$

Notice that out of these four equations, the first two are equivalent to the last two using the assumption that $(H, K)$ is an equivalence of categories.

Clearly DILL-models provide sound models of DILL, but they are in fact also complete [6].

**Remark 1.18.** In [6] the category of DILL-models is defined by requiring that the cartesian category *is* $\mathbb{C}^\star_!$, and not just is equivalent to it. The authors of [6] then argue that DILL provides the internal language of the DILL-models meaning that the category of DILL models is equivalent to the category of DILL theories with translations as morphisms. With our definition of DILL model, we still have a functor constructing the internal language of a model and a functor constructing the classifying model of a theory. For any theory, the internal language of the classifying model is isomorphic to the original theory, and for any model, the classifying model of the internal language is equivalent to the original model.

**Proposition 1.19.** *Given two DILL-models and a morphism between the two corresponding linear categories, there exists an extension of this morphism to a morphism of DILL-models. This extension is unique up to isomorphism.*

*Proof.* The map is up to equivalence given by Lemma 1.16. ☐

We now give two examples of DILL-models. The first is a practical reformulation of the category $\mathbb{C}^\star_!$ and the second (Proposition 1.21) handles a special case in which $\mathbb{C}_!$ is equivalent to $\mathbb{C}^\star_!$.

We now give a different definition of the category $\mathbb{C}^\star_!$.

Objects: Finite vectors of objects from $\mathbb{C}$.

Morphisms: A morphism from $(A_i)_i$ to $(B_j)_j$ is a family of morphisms $(f_j\colon \otimes_i !A_i \to B_j)_j$.

Composition: The composite of $(f_j)_j\colon (A_i)_i \to (B_j)_j$ and $(g_k)_k\colon (B_j)_j \to (C_k)_k$ is

$$\otimes_i !A_i \xrightarrow{\otimes_i \delta_{A_i}} \otimes_i !!A_i \xrightarrow{m} !(\otimes_i !A_i) \xrightarrow{\langle !f_j \rangle_j} \otimes_j (!B_j) \xrightarrow{(g_k)_k} (C_k)_k,$$

where $\langle !f_j \rangle_j$ is the pairing of the functions $!f_j$ defined as

$$!(\otimes_i !A_i) \xrightarrow{d} \otimes_j !(\otimes_i !A_i) \xrightarrow{\otimes_j !f_j} \otimes_j !B_j.$$

Identity: The identity on $(A_i)$ is

$$(\ \otimes_i !A_i \xrightarrow{\pi_i} !A_i \xrightarrow{\epsilon} A_i\ )_i,$$

where $\pi_{i_0}\colon \otimes_i !A_i \to A_{i_0}$ is defined as

$$\otimes_i !A_i \xrightarrow{\otimes_{i \neq i_0} e_{A_i} \otimes id} (\otimes_{i \neq i_0} I) \otimes A_{i_0} \xrightarrow{\cong} A_{i_0}.$$

**Lemma 1.20.** *The description above describes a category. This category is isomorphic to* $\mathbb{C}^\star_!$.

*Proof.* To be able to distinguish the two definitions, for the remainder of this proof we denote by $\mathbb{D}$ the definition just above. We prove that there are bijective correspondences between objects and morphisms of $\mathbb{D}$ and $\mathbb{C}^\star_!$ and that these bijections preserve composition and identity. This way we prove both statements of the lemma simultaneously.

Objects of both $\mathbb{D}$ and $\mathbb{C}^\star_!$ correspond to finite vectors of objects of $\mathbb{C}$. The correspondence on morphisms is given by

$$\mathrm{Hom}_{\mathbb{D}}((A_i)_i, (B_j)_j) \cong \prod_j \mathrm{Hom}_{\mathbb{D}}((A_i)_i, B_j) \cong \prod_j \mathrm{Hom}_{\mathbb{C}}(\otimes_i !A_i, B_j) \cong$$
$$\prod_j \mathrm{Hom}_{\mathbb{C}^!}(\prod_i \delta_{A_i}, \delta_{B_j}) \cong \mathrm{Hom}_{\mathbb{C}^!}(\prod_i \delta_{A_i}, \prod_j \delta_{B_j}).$$

In one direction, this correspondence maps a map in $\mathbb{D}$:

$$(f_j\colon \otimes_i !A_i \to B_j)_j$$

to

$$\otimes_i !A_i \xrightarrow{\otimes_i \delta_{A_i}} \otimes_i !!A_i \xrightarrow{m} !\otimes_i !A_i \xrightarrow{d} \otimes_j !\otimes_i !A_i \xrightarrow{\otimes_j !f_j} \otimes_j !B_j$$

as a map in $\mathbb{C}^!$. Going the other way, given a map of coalgebras $f\colon \otimes_i !A_i \to \otimes_j !B_j$, this map corresponds to $(\epsilon \circ \pi_j \circ f)_j$ in $\mathbb{D}$.

Since these processes are inverses of each other, we know for example that

$$\otimes_i !A_i \xrightarrow{\otimes_i \delta_{A_i}} \otimes_i !!A_i \xrightarrow{m} !\otimes_i !A_i \xrightarrow{d} \otimes_j !\otimes_i !A_i \xrightarrow{\otimes_j !f_j} \otimes_j !B_j \xrightarrow{\pi_j} !B_j \xrightarrow{\epsilon} B_j$$

is simply $f_j$. This allows us to conclude that if we start with two maps $(f_j)\colon (A_i) \to (B_j), (g_k)\colon (B_j) \to (C_k)$ in $\mathbb{D}$, take the corresponding maps in $\mathbb{C}_!^\star$, compose these and then go back into $\mathbb{D}$, we get exactly the composite of $(f_j)$ and $(g_k)$ as defined in $\mathbb{D}$. This proves that the isomorphism preserves composition. It is clear that the isomorphism preserves identity. □

**Proposition 1.21.** *Suppose the linear category* $(\mathbb{C}, !, e, d)$ *has products. Then* $\mathbb{C}_!$ *is equivalent in the category of SMC's to* $\mathbb{C}_!^\star$ *and the usual adjunction between* $\mathbb{C}$ *and* $\mathbb{C}_!$ *is a DILL-model.*

*Proof.* Notice first that there exists a natural isomorphism $\otimes_i !A_i \cong !(\prod_i A_i)$ which is a map of coalgebras, i.e.,

$$\begin{array}{ccccc} !(\prod_i A_i) & & \xrightarrow{\delta_{\prod A_i}} & & !!(\prod_i A_i) \\ \downarrow{\cong} & & & & \downarrow{!(\cong)} \\ \otimes_i !A_i & \xrightarrow{\otimes_i \delta_{A_i}} & \otimes_i !!A_i & \xrightarrow{m} & !(\otimes_i !A_i) \end{array}$$

is commutative. This follows from the Yoneda Lemma and the natural isomorphisms

$$\mathrm{Hom}_{\mathbb{C}^!}(h, \prod \delta_{A_i}) \cong \prod \mathrm{Hom}_{\mathbb{C}^!}(h, \delta_{A_i}) \cong \prod \mathrm{Hom}_{\mathbb{C}}(B, A_i) \cong$$
$$\mathrm{Hom}_{\mathbb{C}}(B, \prod A_i) \cong \mathrm{Hom}_{\mathbb{C}^!}(h, \delta_{\prod A_i}),$$

for $h\colon B \to !B$.

We need to check that the adjunction between $\mathbb{C}$ and $\mathbb{C}_!$ is an SMC adjunction. The SMC structure on $\mathbb{C}_!$ is defined by the product $\delta_A \times \delta_B = \delta_{A\times B}$, and we need to check that the functor $U_!\colon \mathbb{C}_! \to \mathbb{C}$ is strong. But

$$U_!(\delta_A \times \delta_B) = U_!(\delta_{A\times B}) = !(A \times B) \cong !A \otimes !B = U_!(\delta_A) \otimes U_!(\delta_B).$$

The equivalence is given by the obvious inclusion of $\mathbb{C}_!$ into $\mathbb{C}_!^\star$, and the map, that maps $\prod_i \delta_{A_i}$ to $\delta_{\prod A_i}$. Clearly the composition starting at $\mathbb{C}_!$ is the identity. The isomorphism between the other composition and the identity is the isomorphism $\delta_{\prod A_i} \cong \prod \delta_{A_i}$ described above. We need to check that the two functors are in fact strong morphisms of SMC's, and that the two equivalences make the right triangles commute as in the text after Definition 1.17.

The inclusion of $\mathbb{C}_!$ into $\mathbb{C}_!^\star$ is strong by the isomorphism constructed above, and the functor the other way is strong, because $(\prod \delta_{A_i}) \times (\prod \delta_{B_j})$ maps to $\delta_{(\prod A_i)\times(\prod B_j)}$ which is isomorphic to the product of the images (in fact equal up to arrangement of parentheses).

Finally, we will check that the triangles mentioned after Definition 1.17 commute up to isomorphism. By the same remark, it suffices to prove that half the triangles commute, so let us only consider the ones for the inclusion $\mathbb{C}_! \to \mathbb{C}_!^\star$. It is easily seen that these triangles commute. □

## 2 PILL models

**Definition 2.1 (The 2-category of fibred SMC's).** A fibred symmetric monoidal category is a fibration together with a fibred functor

$$\begin{array}{ccc} E \times_B E & \xrightarrow{\otimes} & E \\ & \searrow \quad \swarrow & \\ & B & \end{array}$$

and fibred vertical natural transformations making each fibre into an SMC.

A fibred symmetric monoidal functor is a map of fibrations $(F, K)$:

$$\begin{array}{ccc} \mathbb{E} & \xrightarrow{F} & \mathbb{E}' \\ \downarrow & & \downarrow \\ \mathbb{B} & \xrightarrow{K} & \mathbb{B}' \end{array}$$

together with vertical fibred natural transformations $m, m_I$, such that for each object $\Xi$ in $\mathbb{B}$ the functor $(F_\Xi, m_\Xi, (m_I)_\Xi)$ is an SMC functor. We say that $F$ is strong (strict) if $m, m_I$ are fibred isomorphisms (identities).

A fibred symmetric monoidal natural transformation from $(H, K)$ to $(H', K')$ is a natural transformation of fibred functors $(\alpha,\ \beta)$ as usual, as in

$$\begin{array}{ccc} E & \underset{H'}{\overset{H}{\rightrightarrows}}\ \Downarrow \alpha & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \underset{K'}{\overset{K}{\rightrightarrows}}\ \Downarrow \beta & B' \end{array}$$

such that the usual diagrams are commutative. Notice that these diagrams need not be vertical, for example, the diagram

$$\begin{array}{ccc} H(-) \otimes H(=) & \xrightarrow{m^H} & H((-) \otimes (=)) \\ {\scriptstyle \alpha\otimes\alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ H'(-) \otimes H'(=) & \xrightarrow{m^{H'}} & H'((-) \otimes (=)) \end{array}$$

projects via $p'$ to

$$\begin{array}{ccc} Kp(-) & \xrightarrow{id} & Kp(-) \\ \downarrow{\scriptstyle \beta_{(-)}} & & \downarrow{\scriptstyle \beta_{(-)}} \\ K'p(-) & \xrightarrow{id} & K'p(-) \end{array}$$

since $p(-) = p(=)$ (so the vertical maps are not vertical ...)

Having defined what the 2-category of fibred SMC's is, we can derive the notion of a fibred symmetric monoidal adjunction. We focus on the case of a fibred symmetric adjunction over a specific base category.

The pair of fibred functors $F, G$ in

$$\begin{array}{ccc} \mathbb{D} & \underset{G}{\overset{F}{\leftrightarrows}}\ \bot & \mathbb{E} \\ & \searrow \quad \swarrow & \\ & \mathbb{B} & \end{array}$$

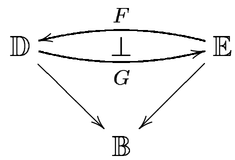
is called a fibred symmetric monoidal adjunction if

- the two fibrations are fibred symmetric monoidal,
- the two functors $F, G$ are fibred symmetric monoidal,

- there exist fibred vertical symmetric monoidal natural transformations $\epsilon\colon FG \Rightarrow id_{\mathbb{D}}, \eta\colon id_{\mathbb{E}} \Rightarrow GF$ such that in each fibre over $\Xi \in \mathbb{B}$, these are counit and unit of the adjunction $F_\Xi \dashv G_\Xi$.

There is a fibred version of Theorem 1.4.

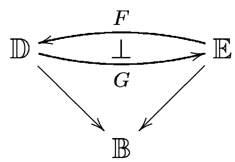
**Theorem 2.2.** *A fibred adjunction*




*between fibred symmetric monoidal fibrations is a symmetric monoidal fibred adjunction iff $F$ is strong.*

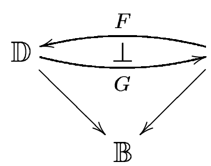
*Proof.* The left adjoint of a fibred symmetric monoidal adjunction is strong since it is strong in each fibre. For the other direction, we notice that the constructions of $m^G, m^G_I$ as described after Theorem 1.4 give us fibred natural transformations, which satisfy the desired properties, since they satisfy them in each fibre. □

**Definition 2.3.** A fibred linear adjunction is a fibred symmetric monoidal adjunction

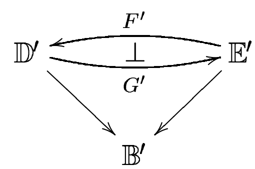



where $\mathbb{D}$ is fibred SMCC and the fibred tensor-product on $\mathbb{E}$ is a fibred cartesian product.

A map of fibred linear adjunctions from




to




is a pair of fibred maps $(H, L)\colon (\mathbb{D} \to \mathbb{B}) \to (\mathbb{D}' \to \mathbb{B}')$ and $(K, L)\colon (\mathbb{E} \to \mathbb{B}) \to (\mathbb{E}' \to \mathbb{B}')$ (over the same map in the base categories) such that $(H, L)$ is a strict map of fibred SMCC's preserving the induced comonad on the nose, and $(K, L)$ is a strong fibred map of SMC's such that the diagrams

$$\begin{array}{ccccc} \mathbb{D} & \xrightarrow{G} & \mathbb{E} & \xrightarrow{F} & \mathbb{D} \\ \downarrow{\scriptstyle H} & & \downarrow{\scriptstyle K} & & \downarrow{\scriptstyle H} \\ \mathbb{D}' & \xrightarrow{G'} & \mathbb{E}' & \xrightarrow{F'} & \mathbb{D}' \end{array}$$

commute up to vertical isomorphism.

A natural transformation of fibred linear adjunctions from

$$((H, L), (K, L)) \text{ to } ((H, L), (K', L))$$

(notice that the $(H, L)$ components of the two maps of fibred linear adjunctions are the same) is a vertical natural transformation from $K$ to $K'$ over $L$.

Again, this may seem a strange definition of natural transformations, but we have chosen this definition to give us a restrictive notion of equivalence of fibred linear adjunctions.

**Definition 2.4.** A fibred linear category is a fibred SMCC with a fibred symmetric monoidal comonad, and fibred symmetric monoidal natural transformations $e, d$ such that for each fibre, the restriction of the data mentioned constitutes a linear category.

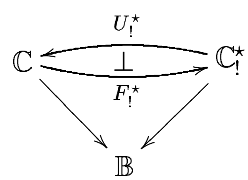
A morphism of fibred linear categories is a strong fibred morphism of SMCC's preserving the comonad structure on the nose as in Definition 1.10. It is called strict, if the functor is a strict fibred symmetric monoidal functor.

**Proposition 2.5.** *There is a forgetful functor from the category of fibred linear adjunctions to the category of fibred linear categories with strict morphisms.*

*Proof.* The proof of Proposition 1.14 clearly generalizes. □

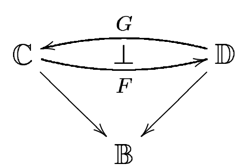
On the other hand, suppose we are given a fibred linear category $\mathbb{C} \to \mathbb{B}$ with comonad !. We can construct the category of coalgebras for the comonad $\mathbb{C}^!$ as having as objects *vertical* maps $A \to !A$ and the rest of the construction as usual. This gives a fibration $\mathbb{C}^! \to \mathbb{B}$. Likewise, we can construct the co-Kleisli fibration $\mathbb{C}_! \to \mathbb{B}$ by taking each fibre to be the co-Kleisli category of the restriction of the comonad, and letting reindexing be the obvious choice. Finally, we can construct $\mathbb{C}_!^\star \to \mathbb{B}$ fibrewise as we did in the Section 1.3.

**Lemma 2.6.** *Given a fibred linear category $\mathbb{C}$ with comonad !, the fibred adjunction*




*is a fibred linear adjunction. This construction extends to a functor which is right inverse to the forgetful functor of Proposition 2.5.*

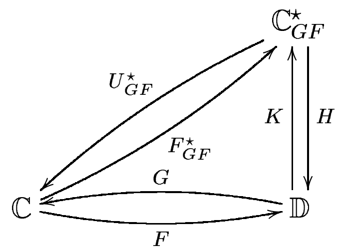
**Definition 2.7.** A PILL-model is a fibred linear adjunction




equivalent to (in the category of fibred linear adjunctions) the fibred linear adjunction induced by the comonad $GF$ as in Lemma 2.6 and such that further

- the category $\mathbb{B}$ is cartesian
- the fibration $\mathbb{C} \to \mathbb{B}$ has a generic object projecting to $\Omega$ in $\mathbb{B}$, and products with respect to projections $\Xi \times \Omega \to \Xi$ in $\mathbb{B}$.

The condition of the fibred linear adjunction being equivalent to the fibred linear adjunction induced by the comonad means that there exists maps $H, K$ fibred over $\mathbb{B}$ as in




such that $H, K$ are strong maps of fibred SMC's and constitute a fibred equivalence, and such that the obvious four triangles commute up to vertical isomorphisms.

**Definition 2.8.** A morphism of PILL-models is a morphism of fibred linear adjunctions such that the SMCC part of the functor preserves generic object, products in the base and products in the fibration.

**Definition 2.9.** A PILL$_Y$-model is a PILL model with a polymorphic fixed point combinator

$$Y\colon \prod \alpha\colon \mathsf{Type}.\,(\alpha \to \alpha) \to \alpha.$$

A morphism of PILL$_Y$-models is a morphism of PILL-models preserving $Y$.

The following Proposition is a trivial generalization of Proposition 1.19.

**Proposition 2.10.** *Given two PILL-models and a morphism of fibred linear categories between the corresponding fibred linear categories preserving generic object, products in the base and the simple products, there exists an extension of this map to a map of PILL-models. The extension is unique up to vertical isomorphism.*

**Lemma 2.11.** *The fibration $\mathbb{C}^\star_! \to \mathbb{B}$ is isomorphic to the fibration obtained by defining each fibre as in Lemma 1.20 and defining reindexing to be the obvious choice.*

*Proof.* The two fibrations are fibrewise isomorphic by Lemma 1.20, and we just need to check that the isomorphism commutes with reindexing, which is obvious. □

**Proposition 2.12.** *Suppose the linear fibration $\mathbb{C} \to \mathbb{B}$ with comonad $!$ has fibrewise products. Then the usual fibred adjunction between $\mathbb{C}$ and $\mathbb{C}_!$ is a fibred linear adjunction, and there exists an equivalence of fibred linear adjunctions between this and the fibred adjunction between $\mathbb{C}$ and $\mathbb{C}^\star_!$.*

*Proof.* The proof of Proposition 1.21 clearly generalizes. □

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