Towards the Theory of Equilogical Spaces

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1 Preface

The goal of this project is to read and understand the main parts of [DS98] and to illustrate the understanding by going into details where the article leaves that to the reader. Specifically, this project is to describe the concept of *equilogical spaces* and to prove that the category they constitute has a certain property (cartesian closure) that is important in Computer Science.

My background for writing this project is a Bachelor's degree in Computer Science and Mathematics and a subsequent introductory course in Category Theory that evolved from [Jaap]. The project will reflect the knowledge that I have gained from it, so the prerequisites for a reader will be a basic knowledge of Category Theory and topology (for example corresponding to [CB]). The "classic" theory of lattices and their connection with topology is new to the me, so there will be an introduction to it, and the subject will be handled in much detail. Furthermore, we will need a few results from Category Theory that was not part of the course.

The rest of the paper is organized as follows: First, we will give a motivation for the equilogical spaces by outlining the problem that comes with considering the category of topological spaces. Next, we will explore some results from lattice theory. Some of these are necessary for the theory of equilogical spaces, while others serve the purpose of giving a broad introduction to the field, while at the same time illustrating understanding of the subject. After this, we will brush up a little category theory, and finally we will get to the goal of the project, namely the category of *equilogical spaces.* We will show that these "fix" the problem that was illustrated in the motivation, and we will also show some other properties that this category possesses. This will be done by considering yet another category, namely the category of modest sets over algebraic lattices.

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A little word on notation: We use calligraphic fonts as in $\mathcal{C}$ to denote non–specific categories, lattices, and topologies. A topological space is then denoted as in [CB]: $(M, \mathcal{T})$, where M is the underlying set and $\mathcal{T}$ is the set of open sets of M. When referring to specific categories, we will use italics, as in TOP_0. We will use subscripts on orderings and bound operators, as in $\leq_{\mathcal{L}}, \bigwedge_{\mathcal{M}}$ so that it is clear which operator we are dealing with. However, when this is obvious, we will omit these subscripts. Furthermore, we will use "ordinary math fonts", as in A, B to denote sets or objects in a category. This font will also be used to denote functors between categories. In the section on category theory, we will freely write $C \in \mathcal{C}$ (or $f \in \mathcal{C}$), and we will mean that C is an object (or f is a morhphism) of $\mathcal{C}$. It will be clear whether we are dealing with objects or arrows, so this will not be confusing. Usually, this notation means that $\mathcal{C}$ is a set (and not a class). Although this might not always be the case, we will use the notation anyway.

I want to thank Lars Birkedal for being an excellent advisor on this project. He has given me invaluable hints and references when I really needed them, but at the same time, he has allowed me to do most of the work on my own, so that I have benefitted optimally from the process.

2 Motivation

When we study the semantics of a programming language, we set up a mathematical model of the different constructs in the language. Since the λ–calculus is a "canonical" programming language, we are interested in modeling it. In chapter 7 in [Jaap], it is shown how one can model the typed λ–calculus in a category if it has the property of being cartesian closed. Intuitively, this works out because a cartesian closed category (ccc) is closed under construction of function spaces, and also the exponentiation in a ccc gives a sound interpretation of currying. In this section we will outline why the category TOP of topological spaces and continuous mappings between them is *not* cartesian closed. This will motivate the concept of equilogical spaces.

The fact that TOP is not cartesian closed is proved in [FB2, Prop 7.1.2], and we will give an intuitive account of the proof here. As one expects, the function space $\mathcal{Y}^{\mathcal{X}}$ between two topological spaces $\mathcal{X} = (X, \mathcal{T}_{\mathcal{X}})$ and $\mathcal{Y} = (Y, \mathcal{T}_{\mathcal{Y}})$ has as its underlying set the set $\mathcal{C}(X, Y)$ of all continuous functions $f : X \to Y$; as for its topology, we do not know very much at a first glance, but what we can say is that if TOP is cartesian closed, then the evaluation map $\varepsilon : \mathcal{Y}^{\mathcal{X}} \times \mathcal{X} \to \mathcal{Y}$ must be continuous (this follows from the fact in [Jaap, p. 64]).

As a concrete counterexample, [FB2] chooses as $\mathcal{X}$ the set $\mathbb{Q}$ of rational numbers with the topology $\mathcal{T}_{\mathbb{Q}}$ inherited from the real numbers. For $\mathcal{Y}$, we choose the unit interval, also with the topology from the reals. Assuming the continuiuty of the evaluation map, one can show that any rational number q has a compact neighborhood V_q, i.e. that $(\mathbb{Q}, \mathcal{T}_{\mathbb{Q}})$ is a locally compact space. The proof of this is lengthy and tedious, so we will skip it here. What we will do, however, is to show that $(\mathbb{Q}, \mathcal{T}_{\mathbb{Q}})$ is *not* locally compact, thus obtaining the desired contradiction.

Suppose that $(\mathbb{Q}, \mathcal{T}_{\mathbb{Q}})$ is locally compact. Then the point 0 has a compact neighborhood V, which contains a basic open subset $]-p, p[$. Let r be an irrational number with $0 < r < p$, and consider the intersection $A = [-r, r] \cap \mathbb{Q}$. Since $[-r, r]$ is closed in the real numbers, A is a closed set in $\mathbb{Q}$ contained in a compact set. It follows by [CB, thm. 6.6] that A itself is compact, so by the definition of A and from the fact that the rational numbers are dense in the reals that

$$A = \bigcup_{q \in \mathbb{Q}, 0 < q < r}]-q, q[$$

It is also obvious that this infinite open covering does not contain a finite subcovering (since this would mean that there were a *single* $q_0 \in \mathbb{Q}$ such that $A =]-q, q[$). This means that A is *not* compact in $(\mathbb{Q}, \mathcal{T}_{\mathbb{Q}})$, and we have arrived at our contradiction.

Another proof of the fact that TOP is not cartesian closed goes like this: Suppose TOP is cartesian closed. Then for all topological spaces S, the product functor $- \times S$ has a right adjoint, so it preserves coequalizers. Since the coequalizers in TOP are precisely the quotient maps, it must be the case that for any topological space S and quotient map $f : Z \to W$, $\langle \mathrm{id}_S, f \rangle$ is again a quotient map. This is not the case, however. Indeed, [DK] gives restrictions on the space S for this to be the case (and $\mathbb{Q}$ does not satisfy them).

There have been two ways of mending this problem. One is to restrict the category, so that we only consider *compactly generated Hausdorff spaces*, and the other solution is to expand the category to the category of *filter spaces*. The first of these is sketched in [ML] but it applies only to topological spaces that have the Haussdorff property, and if we make this restriction, we lose the nice interplay with the lattice theory that is described in the next section. The other alternative has more advantages, but is has the drawback of being fairly complicated, and a thorough desription would require a project the size of this one. There is a treatment of the subject in [Hyl]. The new solution (the equilogical spaces) that Scott proposed in 1996 is in a sense more natural, and it has a nice connection to domain theory. We will get back to this in section 8.

3 Preliminary Lattice Theory

In this section, we will give a gentle introduction to the theory of lattices and order. The goal of all this is to prove that the category *ALat* of *algebraic lattices* and continuous mappings between them is a cartesian closed category (we already see an indication of a connection to topology here). Some of the results in this section are not strictly necessary to establish this, but they are “classics”, and an introduction would not be complete without them. Showing these results will also illustrate understanding of important concepts in lattice theory. We will assume basic knowledge of topology and ordered sets, and we start out with some basic definitions:

Definition 3.1. A *lattice* is a partially ordered set $(P, \leq)$ where every pair x, y of elements have a least upper bound and a greatest lower bound. It is standard notation to denote these $x \vee y$ and $x \wedge y$, respectively. A lattice P is called *complete* if it satisfies that for any subset $S \subseteq P$, the least upper bound $\bigvee S$ and greatest lower bound $\bigwedge S$ of S exist. Note that this implies that a complete lattice has a top element $\top$ and a bottom element $\bot$.

Example 3.2. The standard example of a complete lattice is the powerset lattice: For any set X, consider the powerset $\mathcal{P}(X)$, ordered by set–theoretic inclusion. For any two subsets $A, B \subseteq X$, the least upper bound $A \vee B$ and greatest lower bound $A \wedge B$ are given by $A \cup B$ and $A \cap B$, respectively. This makes $(P, \subseteq)$ a lattice. Similarly, for any family $I \subseteq \mathcal{P}(X)$ of subsets of X, the bounds are also calculated by unions and intersections:

$$\bigvee I = \bigcup_{A_i \in I} A_i \qquad \bigwedge I = \bigcap_{A_i \in I} A_i$$

This means that $\mathcal{P}(A)$ is in fact a complete lattice.

When we want to show that a partially ordered set is a complete lattice, we only need to do half of the work that the definition requires because of the following

Remark 3.3. Assume that $(P, \leq)$ is a partial order, and that $\bigvee S$ exists for all subsets $S \subseteq P$. Then $(P, \leq)$ is a complete lattice.

Proof: First note that our assumptions imply that $\top$ and $\bot$ exist, since we can choose S to be P and $\varnothing$, respectively. For a given $S \subseteq P$, we need to show that $\bigwedge S$ exists; for this we consider the set

$$S_\ell = \{y \in P | \forall x \in S : y \leq x\}$$

We claim that $\bigvee S_\ell$ is the greatest lower bound of S. It is a lower bound, since all $x \in S$ are upper bounds for S_ℓ. It is the greatest lower bound since all lower bounds for S are in S_ℓ. $\square$

Remark 3.4. Note that there is a kind of “idempotency” of the notion of greatest lower bound (glb) (and least upper bound (lub)) in a complete lattice $(P, \leq)$; that is: if you add the glb/lub to a set, then the resulting set has the same glb/lub:

$$\bigwedge (S \cup \{\bigwedge S\}) = \bigwedge S, \qquad \bigvee (S \cup \{\bigvee S\}) = \bigvee S$$

3.1 Lattices and Spaces

We will show a connection between complete lattices and T_0–spaces, and in particular why we can regard complete lattices as topological spaces. First we recall that the *neighborhood filter* $\mathcal{T}(x)$ for a point $x \in M$ in a topological space $(M, \mathcal{T})$ is defined by

$$\mathcal{T}(x) = \{O \in \mathcal{T} | x \in O\}$$

We can use the neighborhood filter to construct an ordering on a topological space.

Definition 3.5. Let $(M, \mathcal{T})$ be a topological space. The *specialization ordering* $\leq_{\mathcal{T}}$ on M is defined by

$$x \leq_{\mathcal{T}} y \iff \mathcal{T}(x) \subseteq \mathcal{T}(y)$$

for all $x, y \in M$.

Clearly, this is a preorder on the points of the topological space. In general, it is not a partial order (consider for example the trivial topology $\mathcal{T} = \{M, \varnothing\}$, which makes the specialization ordering equal to $M \times M$). Antisymmetry can be obtained if we cut down to considering a special class of topological spaces:

Definition 3.6. A topological space $(M, \mathcal{T})$ is a T_0*–space* if for every two distinct points, there is an open set which contains one, but not the other. The category of such topological spaces and continuous functions between them is denoted TOP_0.

Remark 3.7. Remark that the condition in 3.6 is weaker than the "separation axioms" given in [CB], section 5.1. It is also easy to see that T_0–spaces are partially ordered by the specialization ordering. Also note that a topological space $(M, \mathcal{T})$ is a T_0–space if and only the implication

$$\mathcal{T}(x) = \mathcal{T}(y) \Longrightarrow x = y$$

holds for all $x, y \in M$.

In a partial order $(P, \leq)$, a subset $S \subseteq P$ is called *upwards closed* provided $x \in S$ and $x \leq y$ always imply $y \in S$. It is a simple exercise to show that the upwards closed subsets of a partial order $(P, \leq)$ constitute a topology on P (and it also follows from the proof of the upcoming theorem 3.10), and futhermore that the specialization ordering induced by this topology is precisely the same ordering as the one we started out with. There is a similar interplay between complete lattices and T_0–spaces, which we will see in a moment. To ease the definition of the topology that we will endow complete lattices with, we define a predicate on subsets of a complete lattice.

Definition 3.8. Let $(L, \leq)$ be a complete lattice. We say that a subset $U \subseteq L$ satisfies the condition (C) if

(C) For every subset $S \subseteq P$ with $\bigvee S \in U$, there exists a *finite* subset $S_0 \subseteq S$ such that $\bigvee S_0 \in U$.

We are now ready to define a topology on a complete lattice.

Definition 3.9. Let $\mathcal{L} = (L, \leq)$ be a complete lattice. We define the *Σ–topology* on L to be the set of all upwards closed subsets $U \subseteq L$ that that satisfy the condition (C). We denote this topology $\Sigma_{\mathcal{L}}$.

The following theorem gives the connection between complete lattices and T_0–spaces.

Theorem 3.10. *Let $\mathcal{L} = (L, \leq_{\mathcal{L}})$ be a complete lattice. Then the structure $(L, \Sigma_{\mathcal{L}})$ is a T_0–space, and the corresponding specialization ordering coincides with $\leq_{\mathcal{L}}$.*

Proof: Strictly speaking, we have not even established that $\Sigma_{\mathcal{L}}$ is a topology on L yet, so we start off with that. Obviously, both L and $\varnothing$ are upwards closed and satisfy the condition (C). Next, let

$$U = U_1 \cap \cdots \cap U_n$$

be a finite intersection of sets U_i from $\Sigma_{\mathcal{L}}$. We are to check that U is upwards closed and satisfies condition (C). So assume first that $x \in U$ and that $x \leq y$. Since all the U_i's are upwards closed, we have that $y \in U_i$ for all $i \in \{1, \ldots, n\}$. This means that $y \in U$, so U is upwards closed. Next, assume that for a subset $S \subseteq L$, we have that $\bigvee S \in U$. This means that $\bigvee S \in U_i$ for all $i \in \{1, \ldots, n\}$, so there exist finite sets $S_1, \ldots, S_n$, such that $\bigvee S_i \in U_i$ for $i \in \{1, \ldots, n\}$. Now, let

$$S_0 := \bigcup_{i=1}^{n} S_i$$

and note that S_0 is finite. Clearly, for all $i \in \{1, \ldots, n\}$, we have that $\bigvee S_i \leq \bigvee S_0$, and since U_i is upwards closed, we get that $\bigvee S_0 \in U_i$ for all $i \in \{1, \ldots, n\}$; thus $\bigvee S_0 \in U$. This means that U satisfies the condition (C), so $\Sigma_{\mathcal{L}}$ is closed under finite intersection. To show that it is a topology, it remains to show that it is stable under arbitrary unions. Therefore, let

$$V = \bigcup_{i \in I} V_i$$

be a union of sets V_i from $\Sigma_{\mathcal{L}}$. Then V is upwards closed, since if $x \in V$ and $x \leq y$, then $x \in V_{i_0}$ (which is upwards closed) for an $i_0 \in i$ and this means that

$y \in V_{i_0} \subseteq V$. To show that V satisfies the condition (C), we assume that we have a subset $S \subseteq L$ with $\bigvee S \in V$. Then there exists an $i_0 \in I$ such that $\bigvee S \in V_{i_0}$, which in turn means that there exists a finite $S_0 \subseteq S$ with $\bigvee S_0 \in V_{i_0} \subseteq V$, i.e. V satisfies condition (C), and therefore, it is in $\Sigma_{\mathcal{L}}$. That is, we have shown that $\Sigma_{\mathcal{L}}$ is a topology on L.

Before we embark on the proof of the T_0–property, we consider an auxiliary class of sets. For a point $p \in L$, we define

$$\overline{p} = \{q \in L | q \nleq p\}$$

We wish to show that $\overline{p}$ is open. It is upwards closed, since if $q \in \overline{p}$ and $q \leq s$, then $s \in \overline{p}$, since if $s \leq p$, then $q \leq s \leq p$, which is a contradiction. To show that $\overline{p}$ satisfies condition (C), assume $\bigvee S \in \overline{p}$ (i.e. $\bigvee S \nleq p$) for a subset $S \subseteq L$. Then there exists an $s_0 \in S$ such that $s_0 \nleq p$, since otherwise p would be an upper bound for S so that $\bigvee S \leq p$. This means that $\bigvee\{s_0\} \in \overline{p}$, and $\overline{p}$ satisfies the condition (C) and is open.

We now return to the T_0–property. Let two distinct points $x, y \in L$ be given. There are two cases: (i): x and y are incomparable, and (ii): Either $x \leq y$ or $y \leq x$. If they are incomparable, then we have that $x \notin \overline{x}$, but $y \in \overline{x}$. Otherwise, if $x \leq y$ (the case $y \leq x$ is similar), then we must have that $y \nleq x$ since $\leq$ is antisymmetric and x and y are distinct. This means that $y \in \overline{x}$, whereas $x \notin \overline{x}$. Thus, $(L, \Sigma_{\mathcal{L}})$ is a T_0–space. We now consider the specialization ordering $\leq_\Sigma$ corresponding to this topological space. It remains to be shown that for any two points $x, y \in L$, we have that

$$x \leq_\Sigma y \iff x \leq_{\mathcal{L}} y$$

Assume that $x \leq_{\mathcal{L}} y$, and let $x \in U$, where U is open in the Σ–topology. Then U is upwards closed, and $y \in U$ so $x \leq_\Sigma y$. On the other hand, assume that $x \leq_\Sigma y$, i.e. all open sets that contain x, will also contain y. If $x \nleq_{\mathcal{L}} y$, then we have that $x \in \overline{y}$, but $y \notin \overline{y}$, which is a contradiction. This completes the proof of the theorem. □

With this theorem, we can treat all complete lattices as T_0–spaces. As of yet, we have not considered what the closed sets in $(L, \Sigma_{\mathcal{L}})$ are. An upcoming characterization deals with this, but we need some more terminology:

Definition 3.11. A *directed* subset of a partially ordered set $(P, \leq)$ is a subset $D \subseteq P$ such that every finite subset of D has an upper bound in D. We say that a subset $A \subseteq L$ is *closed under directed sups* if for any directed set $D \subseteq A$, we have $\bigvee D \in A$.

Theorem 3.12. *Let $\mathcal{L} = (L, \leq)$ be a complete lattice. A subset $A \subseteq L$ is closed in the $\Sigma_{\mathcal{L}}$–topology if and only if it is downwards closed and closed under directed sups.*

Proof: Let A be a closed set in the $\Sigma_{\mathcal{L}}$–topology (so that $\complement A$ is open). If $x \in A$ and $z \leq x$, then we must have that $z \in A$, for if $z \in \complement A$, then the upwards closedness of $\complement A$ would imply $x \in \complement A$, which is a contradiction. Therefore, A is downwards closed. Next, assume that $D \subseteq A$ is a directed set, and consider the element $\bigvee D$. If $\bigvee D \in \complement A$, then there would exist a finite subset $D_0 \subseteq D$ with $\bigvee D_0 \in \complement A$ because $\complement A$ satisfies the condition (C). But since D is directed, we have that $\bigvee D_0 \in D \subseteq A$, which is a contradiction. This means that $\bigvee D \in A$, so A is closed under directed sups.

Conversely, assume that A is downwards closed and closed under directed sups. We are to show that $\complement A$ is open. It follows by an easy argument dual to the one above that $\complement A$ is upwards closed. To show that $\complement A$ satisfies the condition (C), let $S \subseteq L$ be such that $\bigvee S \in \complement A$, and assume for the sake of contradiction that for all finite subsets $S_0 \subseteq S$, we have that $\bigvee S_0 \in A$. Consider the set

$$D = \left\{ \bigvee S_0 | S_0 \subseteq S \text{ is finite} \right\}$$

We claim that D is directed. If $s_1, \ldots, s_n \in D$, then there are finite sets $S_1, \ldots, S_n$ such that $s_i = \bigvee S_i$ for each $i \in \{1, \ldots, n\}$. Then the set $S_0 = S_1 \cup \cdots \cup S_n$ is finite, and $s_0 = \bigvee S_0 \in D$ is an upper bound for $s_1, \ldots, s_n$. By assumption, we have that $D \subseteq A$, and since A is closed under directed sups, we get that $\bigvee D \in A$. But

$$\bigvee D = \bigvee \left\{ \bigvee S_0 | S_0 \subseteq S \text{ is finite} \right\} = \bigvee S$$

and we have arrived at our contradiction. $\square$

Remark 3.13. Note that we can extract the following dualities in a complete lattice from the preceeding proof:

- A upwards closed $\iff$ $\complement A$ is downwards closed.
- A satisfies the condition (C) $\iff$ $\complement A$ is closed under directed sups.

To end this section, we note that continuity in the Σ–topology implies monotonicity:

Corollary 3.14. *Let $\mathcal{L} = (L, \leq_{\mathcal{L}})$ and $\mathcal{M} = (M, \leq_{\mathcal{M}})$ be complete lattices, and assume that $f : (L, \Sigma_{\mathcal{L}}) \to (M, \Sigma_{\mathcal{M}})$ is continuous. Then f is monotone.*

Proof: For two points $x, y \in L$ we assume that $x \leq_{\mathcal{L}} y$, and we are to prove that $f(x) \leq_{\mathcal{M}} f(y)$. By theorem 3.10, it suffices to show that $\Sigma_{\mathcal{M}}(f(x)) \subseteq \Sigma_{\mathcal{M}}(f(y))$, i.e.: all open sets that contain $f(x)$ will also contain $f(y)$. Let $U \subseteq M$ be an open set with $f(x) \in U$. This means that $x \in f^{-1}(U)$. Since f is continuous, we have that $f^{-1}(U)$ is open in the $\Sigma_{\mathcal{L}}$ topology, so in particular, it is upwards closed. This means that $y \in f^{-1}(U)$, so $f(y) \in U$, as desired. $\square$

4 Categorical Properties of Complete Lattices

Definition 4.1. The category *CLat* is defined as follows:

- *Objects* are complete lattices.
- *Morphisms* are functions between lattices that are continuous under the Σ–topology.

We will study this category in this section. The goal is to prove that it is a cartesian closed category. The results that we will prove in this section will be based on lattice–theoretic proofs. Therefore, we will need a more "lattice–theoretic" definition of continuity (a function is only a morphism in *CLat* if it is continuous).

Theorem 4.2. *Let $\mathcal{L} = (L, \leq_{\mathcal{L}})$ and $\mathcal{M} = (M, \leq_{\mathcal{M}})$ be complete lattices, and let $f : L \to M$ be a function. Then $f : (L, \Sigma_{\mathcal{L}}) \to (M, \Sigma_{\mathcal{M}})$ is continuous if and only if*

$$f(\bigvee_{\mathcal{L}} D) = \bigvee_{\mathcal{M}} f(D)$$

for all directed $D \subseteq L$.

Proof: Suppose $f(\bigvee_{\mathcal{L}} D) = \bigvee_{\mathcal{M}} f(D)$ for all directed subsets $D \subseteq L$. We first wish to show that f is monotone when considered as a function $f : \mathcal{L} \to \mathcal{M}$. Assume therefore that $x \leq_{\mathcal{L}} y$, and note that this makes $\{x, y\}$ a directed subset of L. By assumption, we therefore have that

$$f(y) = f(\bigvee_{\mathcal{L}}\{x, y\}) = \bigvee_{\mathcal{M}} f(\{x, y\}) = \bigvee_{\mathcal{M}}\{f(x), f(y)\} = f(x) \vee f(y)$$

This means that $f(x) \leq f(y)$, so F is monotone. To show that it is continuous, we show that the inverse image of any closed subset in M is closed in L. So let $A \subseteq M$ be a closed subset of M, and consider $f^{-1}(A)$. We need to show that it is downwards closed and closed under directed sups. Let $x \in f^{-1}(A)$, and $z \leq x$. This means that $f(x) \in A$, and since f is monotone, we get that $f(z) \leq f(x)$, which, since A is downwards closed, in turn means that $f(z) \in A$, i.e. $z \in f^{-1}(A)$, so $f^{-1}(A)$ is downwards closed. For the other condition, assume that $D \subseteq f^{-1}(A)$ is a directed subset of L (this means that $f(D) \subseteq A$). We are to show that $\bigvee_{\mathcal{L}} D \in f^{-1}(A)$, or equivalently, that $f(\bigvee_{\mathcal{L}} D) \in A$. By our assumption about f, we have that

$$f(\bigvee_{\mathcal{L}} D) = \bigvee_{\mathcal{M}} f(D)$$

Using the monotonicity of f, it is easy to see that the set $f(D) \subseteq M$ is directed, and since $f(D) \subseteq A$ and A is closed under directed sups, we get that $f(\bigvee_{\mathcal{L}} D) \in A$, as desired.

To show the "only if" part, assume that $f : (L, \Sigma_{\mathcal{L}}) \to (M, \Sigma_{\mathcal{M}})$ is continuous, and let $D \subseteq L$ is a directed set. By corollary 3.14, we get that f is monotone, so it is obvious that $\bigvee_{\mathcal{M}} f(D) \leq_{\mathcal{M}} f(\bigvee_{\mathcal{L}} D)$. Assume $\bigvee_{\mathcal{M}} f(D) \not\leq_{\mathcal{M}} f(\bigvee_{\mathcal{L}} D)$. Then we can use the proof of theorem 3.10 to conclude that there exists a closed set A $(= \complement\overline{f(\bigvee_{\mathcal{L}} D)})$ such that $\bigvee_{\mathcal{M}} f(D) \notin A$, whereas $f(\bigvee_{\mathcal{L}} D) \in A$. By monotonicity of f and directedness of D, we get that $f(D)$ is directed. Furthermore, we have that $\bigvee_{\mathcal{L}} D \in f^{-1}(A)$, and since f is continuous, $f^{-1}(A)$ is closed, and in particular, downwards closed. This means that $D \subseteq f^{-1}(A)$, so $f(D) \subseteq A$. But A is closed under directed sups, so $\bigvee_{\mathcal{M}} f(D) \in A$, and we have arrived at our contradiction. This completes the proof. □

Remark 4.3. In [DS98], the result in 4.2 is mentioned only briefly, and there is no treatment of the closed set in the space $(L, \Sigma_{\mathcal{L}})$. I haven't been able to prove the result without taking advantage of the closed sets. Furthermore, if one looks in [GG, p. 113], one will find that they also use the closed sets to prove the equivalence.

Using our newly obtained characterization of morphisms in *CLat*, we move further towards showing that our category is cartesian closed. For this, we first show some convienient properties of the set of continuous functions between two complete lattices.

Definition 4.4. Let $\mathcal{L} = (L, \leq_{\mathcal{L}})$ and $\mathcal{M} = (M, \leq_{\mathcal{M}})$ be complete lattices, and consider the set $(\mathcal{L} \to \mathcal{M})$ of continuous functions from $\mathcal{L}$ to $\mathcal{M}$. We define an ordering $\leq_{(\mathcal{L}\to\mathcal{M})}$ on this set by

$$f \leq_{(\mathcal{L}\to\mathcal{M})} g \iff \forall x \in L : f(x) \leq_{\mathcal{M}} g(x)$$

It is easy to see that this gives a partial ordering on $(\mathcal{L} \to \mathcal{M})$. It is called the *pointwise ordering* of functions.

The following theorem shows that the partial order $((\mathcal{L} \to \mathcal{M}), \leq_{(\mathcal{L}\to\mathcal{M})})$ is in fact a complete lattice. This is theorem 2.16 in [DS98] and it is also noted on p. 33 in [DP].

Theorem 4.5. *For two complete lattices $\mathcal{L} = (L, \leq_{\mathcal{L}})$ and $\mathcal{M} = (M, \leq_{\mathcal{M}})$, the function space $(\mathcal{L} \to \mathcal{M})$ is a complete lattice under $\leq_{(\mathcal{L}\to\mathcal{M})}$.*

Proof: By remark 3.3, we just need to find least upper bounds for a given subset $F = \{f_i | i \in I\} \subseteq (\mathcal{L} \to \mathcal{M})$. For this, we define the function $(\bigvee_{(\mathcal{L}\to\mathcal{M})} F) : L \to M$ by

$$(\bigvee_{(\mathcal{L}\to\mathcal{M})} F)(x) = \bigvee_{\mathcal{M}} \{f(x) | f \in F\}$$

We claim that this is the least upper bound for F in $((\mathcal{L} \to \mathcal{M}), \leq_{(\mathcal{L}\to\mathcal{M})})$. For this we need to show that

- $(\bigvee_{(\mathcal{L}\to\mathcal{M})} F)$ is well–defined.
- $(\bigvee_{(\mathcal{L}\to\mathcal{M})} F) \in (\mathcal{L} \to \mathcal{M})$, i.e. $(\bigvee_{(\mathcal{L}\to\mathcal{M})} F)$ is continuous.
- $(\bigvee_{(\mathcal{L}\to\mathcal{M})} F)$ is an upper bound for F and the least element having this property.

$(\bigvee_{(\mathcal{L}\to\mathcal{M})} F)$ is well–defined since $\bigvee_{\mathcal{M}}\{f(x)|f \in F\}$ is defined for all x by completeness of $\mathcal{M}$. To show that it is continuous, we show that it preserves least upper bounds of directed sets. So, let $D \subseteq L$ be a directed set. We then have

$$\begin{array}{rcll}
(\bigvee_{(\mathcal{L}\to\mathcal{M})} F)(\bigvee_{\mathcal{L}} D) & = & \bigvee_{\mathcal{M}} \{f(\bigvee_{\mathcal{L}} D) | f \in F\} & \text{By definition} \\
& = & \bigvee_{\mathcal{M}} \{\bigvee_{\mathcal{M}} f(D) | f \in F\} & \text{Each } f \in F \text{ is continuous} \\
& = & \bigvee_{\mathcal{M}} (\bigvee_{(\mathcal{L}\to\mathcal{M})} F)(D) & \text{By definition}
\end{array}$$

This means that $(\bigvee_{(\mathcal{L}\to\mathcal{M})} F) \in (\mathcal{L} \to \mathcal{M})$, as desired. The last thing we need to show is that we have indeed found the least upper bound of F in $(\mathcal{L} \to \mathcal{M})$. First we show that it is an upper bound, so let $f \in F$. We are then to show that $f \leq_{(\mathcal{L}\to\mathcal{M})} (\bigvee_{(\mathcal{L}\to\mathcal{M})} F)$, i.e. that we for all $x \in L$ have

$$f(x) \leq_{\mathcal{M}} (\bigvee_{(\mathcal{L}\to\mathcal{M})} F)(x) = \bigvee_{\mathcal{M}}\{f(x)|f \in F\}$$

But since $f \in F$, this is obvious. To show that $(\bigvee_{(\mathcal{L}\to\mathcal{M})} F)$ is the least upper bound, let $g : L \to M$ be a continuous function such that $f \leq_{(\mathcal{L}\to\mathcal{M})} g$ for all $f \in F$. We are then to show that $(\bigvee_{(\mathcal{L}\to\mathcal{M})} F) \leq_{(\mathcal{L}\to\mathcal{M})} g$, so let an $x \in L$ be given. As g is an upper bound for F, we have that $g(x) \leq_{\mathcal{M}} f(x)$, i.e. $g(x)$ is an upper bound in $\mathcal{M}$ for $\{f(x)|f \in F\}$. By definition, this means that

$$(\bigvee_{(\mathcal{L}\to\mathcal{M})} F)(x) = \bigvee_{\mathcal{M}}\{f(x)|f \in F\} \leq_{\mathcal{M}} g(x)$$

All in all, we have shown that $(\bigvee_{(\mathcal{L}\to\mathcal{M})} F)$ is the least upper bound of F in $((\mathcal{L} \to \mathcal{M}), \leq_{(\mathcal{L}\to\mathcal{M})})$, and the theorem is proved. □

This result establishes that *CLat* is closed under the construction of function spaces; intuitively this is a necessary condition of cartesian closure. To move further towards our goal of this section, we must establish that *CLat* has a terminal object and binary products (which implies that it has all finite products). The terminal object of *CLat* is just the lattice $\{\star\}$. To show that it has binary products, we define product lattices in an obvious way:

Definition 4.6. Let $\mathcal{L} = (L, \leq_{\mathcal{L}})$ and $\mathcal{M} = (M, \leq_{\mathcal{M}})$ be lattices. The *product lattice* $\mathcal{L} \times \mathcal{M}$ has as underlying set the cartesian product $L \times M$ and as ordering the componentwise ordering $\leq_{\mathcal{L}\times\mathcal{M}}$ defined as

$$(x, y) \leq_{\mathcal{L}\times\mathcal{M}} (x', y') \iff x \leq_{\mathcal{L}} x' \text{ and } y \leq_{\mathcal{M}} y'$$

It is easy to see that greatest lower bounds and least upper bounds are computed componentwise (i.e. by projecting to the coordinates and then taking glb and lub), and that if $\mathcal{L}$ and $\mathcal{M}$ are complete lattices, then so is $\mathcal{L}\times\mathcal{M} = (L\times M, \leq_{\mathcal{L}\times\mathcal{M}})$.

We are going to show cartesian closure of *Clat* by showing that a certain adjunction always exists. The following lemma will be helpful in this regard. It is a slightly specialized version of exercise 3.8 in [DP].

Lemma 4.7. *Let* $\mathcal{L} = (L, \leq_{\mathcal{L}})$*,* $\mathcal{M} = (M, \leq_{\mathcal{M}})$*, and* $\mathcal{N} = (N, \leq_{\mathcal{N}})$ *be complete lattices, and let* $\varphi : L \times M \to N$ *be a map.*

(a) For given $x \in L$ *and* $y \in M$*, we define* $\varphi_x : M \to N$ *and* $\varphi_y : L \to N$ *by*

$$\varphi_x(y) = \varphi(x, y) \qquad \text{and} \qquad \varphi_y(x) = \varphi(x, y)$$

Then φ *is continuous if and only if* φ_x *and* φ_y *are continuous for all* $x \in L$ *and* $y \in M$*.*

(b) If φ *is continuous, then the function* $\Phi : L \to (M \to N)$ *defined by*

$$(\Phi(x))(y) = \varphi(x, y)$$

is well–defined and continuous.

Moreover, the two operations $m_{\mathcal{L},\mathcal{N}} : (\mathcal{L} \times \mathcal{M} \to \mathcal{N}) \to (\mathcal{L} \to (\mathcal{M} \to \mathcal{N}))$ *and* $m^{-1}_{\mathcal{L},\mathcal{N}} : (\mathcal{L} \to (\mathcal{M} \to \mathcal{N})) \to (\mathcal{L} \times \mathcal{M} \to \mathcal{N})$ *defined by*

$$m_{\mathcal{L},\mathcal{N}}(\varphi) = \Phi \quad \text{and} \quad m^{-1}_{\mathcal{L},\mathcal{N}}(\Phi) = \varphi$$

are mutually inverse, so we have a bijection $(\mathcal{L} \times \mathcal{M} \to \mathcal{N}) \cong (\mathcal{L} \to (\mathcal{M} \to \mathcal{N}))$*.*

It should be noted that when we for example say that "$\varphi_x : M \to N$ is continuous", we mean that it is continuous when regarded as a mapping between the corresponding lattices $\mathcal{M}$ and $\mathcal{N}$.

PROOF: *(a)* Assume first that $\varphi : L \times M \to N$ is continuous. We just show that φ_y is continuous for all $y \in M$, since the proof of continuity of φ_x is completely symmetric. Let $y \in M$ be given and let $D \subseteq L$ be a directed set. It is easy to see that this makes the set $D_y = \{(x, y) | x \in D\} \subseteq L \times M$ a directed set in $\mathcal{L} \times \mathcal{M}$. We then have that

$$\begin{aligned}
\varphi_y(\textstyle\bigvee_{\mathcal{L}} D) &= \varphi(\textstyle\bigvee_{\mathcal{L}} D, y) && \text{By definition of } \varphi_y \\
&= \varphi(\textstyle\bigvee_{\mathcal{L}\times\mathcal{M}} D_y) && \text{Easy calulation: } \textstyle\bigvee_{\mathcal{L}\times\mathcal{M}} D_y = (\bigvee_{\mathcal{L}} D, y) \\
&= \textstyle\bigvee_{\mathcal{N}} \varphi(D_y) && \text{Since } \varphi \text{ is continuous and } D_y \text{ directed} \\
&= \textstyle\bigvee_{\mathcal{N}} \varphi(D, y) && \text{By definition of } D_y \\
&= \textstyle\bigvee_{\mathcal{N}} \varphi_y(D) && \text{By definition of } \varphi_y
\end{aligned}$$

Since both $\mathcal{L}$ and $\mathcal{N}$ are complete lattices, we can conclude that φ_y is continuous by theorem 4.2. To show the reverse direction, assume that φ_x and φ_y are continuous for all $x \in L, y \in M$. We show first that φ is monotone, so let $(x, y) \leq_{\mathcal{L}\times\mathcal{M}} (x', y')$. Then because φ_x and $\varphi_{y'}$ are continuous (and thereby monotone by theorem 3.14), we have that

$$\varphi(x, y) = \varphi_x(y) \leq \varphi_x(y') = \varphi(x, y') = \varphi_{y'}(x) \leq \varphi_{y'}(x') = \varphi(x', y')$$

This shows that φ is monotone. To show that it is also continuous, let $D \subseteq L \times M$ be a directed set in $\mathcal{L} \times \mathcal{M}$, and let $D_{\mathcal{L}}$ and $D_{\mathcal{M}}$ be the projections of D on L and M, respectively (so that $\bigvee_{\mathcal{L}\times\mathcal{M}} D = (\bigvee_{\mathcal{L}} D_{\mathcal{L}}, \bigvee_{\mathcal{M}} D_{\mathcal{M}})$).We then have the following calculation:

$$\begin{aligned}
\varphi(\textstyle\bigvee_{\mathcal{L}\times\mathcal{M}} D) &= \varphi(\textstyle\bigvee_{\mathcal{L}} D_{\mathcal{L}}, \bigvee_{\mathcal{M}} D_{\mathcal{M}}) && (1) \\
&= \varphi_{\bigvee_{\mathcal{M}} D_{\mathcal{M}}}(\textstyle\bigvee_{\mathcal{L}} D_{\mathcal{L}}) && \\
&= \textstyle\bigvee_{\mathcal{N}} \varphi_{\bigvee_{\mathcal{M}} D_{\mathcal{M}}}(D_{\mathcal{L}}) && (2) \\
&= \textstyle\bigvee_{\mathcal{N}} \varphi(D_{\mathcal{L}}, \bigvee_{\mathcal{M}} D_{\mathcal{M}}) && \\
&= \textstyle\bigvee_{\mathcal{N}} \{\varphi(x, \bigvee_{\mathcal{M}} D_{\mathcal{M}}) | x \in D_{\mathcal{L}}\} && \\
&= \textstyle\bigvee_{\mathcal{N}} \{\varphi_x(\bigvee_{\mathcal{M}} D_{\mathcal{M}}) | x \in D_{\mathcal{L}}\} && \\
&= \textstyle\bigvee_{\mathcal{N}} \{\bigvee_{\mathcal{N}} \varphi_x(D_{\mathcal{M}}) | x \in D_{\mathcal{L}}\} && (3) \\
&= \textstyle\bigvee_{\mathcal{N}} \{\bigvee_{\mathcal{N}} \varphi(x, D_{\mathcal{M}}) | x \in D_{\mathcal{L}}\} && \\
&= \textstyle\bigvee_{\mathcal{N}} \{\bigvee_{\mathcal{N}} \{\varphi(x, y) | y \in D_{\mathcal{M}}\} | x \in D_{\mathcal{L}}\} && \\
&= \textstyle\bigvee_{\mathcal{N}} \{\bigvee_{\mathcal{N}} \{\varphi(x_2, y_1) | \exists x_1 : (x_1, y_1) \in D\} | \exists y_2 : (x_2, y_2) \in D\} && (4)
\end{aligned}$$

Some of these equations deserve an explanation. They are marked by numbers and explained here:

(1) By the remark above
(2) By continuity of $\varphi_{\bigvee_{\mathcal{M}} D_{\mathcal{M}}}$
(3) By continuity of φ_x for $x \in D_{\mathcal{L}}$
(4) This equation holds because the statement $y \in D_{\mathcal{M}}$ means $\exists x \in L : (x, y) \in D$, and similarly for the statement $x \in D_{\mathcal{L}}$.

Now the set D is directed, so every time we choose $d_1 = (x_1, y_1)$ and $d_2 = (x_2, y_2)$ from D, there exists a point $e = (e_1, e_2) \in D$ such that $d_1, d_2 \leq_{\mathcal{L}\times\mathcal{M}} e$, in particular: $x_2 \leq_{\mathcal{L}} e_2$ and $y_1 \leq_{\mathcal{M}} e_1$. Since φ is monotone, this means that we can rewrite this last "double lub" into a single one:

$$\begin{array}{rcl}\varphi(\bigvee_{\mathcal{L}\times\mathcal{M}} D) & = & \bigvee_{\mathcal{N}}\{\bigvee_{\mathcal{N}}\{\varphi(x_2,y_1)|\exists x_1 : (x_1,y_1)\in D\}|\exists y_2 : (x_2,y_2)\in D\} \\ & = & \bigvee_{\mathcal{N}}\{\varphi(e_1,e_2)|(e_1,e_2)\in D\} \\ & = & \bigvee_{\mathcal{N}}\varphi(D)\end{array}$$

This means that φ is continuous, as desired.

(b): Assume that $\varphi : \mathcal{L}\times\mathcal{M}\to\mathcal{N}$ is continuous, and define Φ as in the statement of the lemma. We need to show that it is (i) well–defined, i.e. that $\Phi(x) : \mathcal{M}\to\mathcal{N}$ is continuous for all $x\in\mathcal{L}$, and (ii) continuous as a mapping between the two complete lattices $\mathcal{L}$ and $(\mathcal{M}\to\mathcal{N})$. Since φ is continuous, we have just seen that $\varphi_x : \mathcal{M}\to\mathcal{N}$ is continuous. Since $\Phi(x)=\varphi_x$, we get that Φ is well–defined. To show that Φ is continuous, we let $D\subseteq L$ be a directed set (and let D_y be defined as above). We are to show that

$$\Phi(\bigvee_{\mathcal{L}} D) = \bigvee_{(\mathcal{L}\to\mathcal{M})}\Phi(D) \qquad (:\mathcal{M}\to\mathcal{N})$$

We show that the two functions coincide for all values in $\mathcal{M}$, so let a $y\in\mathcal{M}$ be given. We then have that

$$\begin{array}{rcll}(\Phi(\bigvee_{\mathcal{L}} D))(y) & = & \varphi(\bigvee_{\mathcal{L}} D, y) & \text{By definition of } \Phi \\ & = & \varphi(\bigvee_{\mathcal{L}\times\mathcal{M}} D_y) & \\ & = & \bigvee_{\mathcal{N}}\varphi(D_y) & \text{By continuity of } \varphi \\ & = & \bigvee_{\mathcal{N}}\varphi(D,y) & \\ & = & \bigvee_{\mathcal{N}}((\Phi(D))(y)) & \text{By definition of } \Phi \\ & = & \bigvee_{\mathcal{N}}\{f(y)|f\in\Phi(D)\} & \\ & = & \left(\bigvee_{(\mathcal{M}\to\mathcal{N})}\Phi(D)\right)(y) & \text{By defintion of lub in } (\mathcal{L}\to\mathcal{M})\end{array}$$

This shows that Φ is continuous.

The proof that the two operations $m_{\mathcal{L},\mathcal{N}}$ and $m^{-1}_{\mathcal{L},\mathcal{N}}$ are mutually inverse is straightforward, and we will not do it here. □

We are now ready to state and prove the main theorem in this section:

Theorem 4.8. *The category* CLat *of complete lattices and continuous mappings between them is a cartesian closed category.*

Proof: We have already seen that *CLat* has all finite products, so we need to show that for any object $\mathcal{M}$ in *CLat*, the product functor $(-)\times\mathcal{M}$ defined by

$$\mathcal{L}\mapsto\mathcal{L}\times\mathcal{M}$$

has a right adjoint. Naturally, we claim that the function space functor $(\mathcal{M}\to -)$ given by

$$\mathcal{N}\mapsto(\mathcal{M}\to\mathcal{N})$$

is the desired right adjoint. For this we need to show that for two objects $\mathcal{L}, \mathcal{N} \in CLat$, there is a natural bijection between the hom–sets

$$CLat(\mathcal{L} \times \mathcal{M}, \mathcal{N}) \xrightarrow{m_{\mathcal{L},\mathcal{N}}} CLat(\mathcal{L}, (\mathcal{M} \to \mathcal{N}))$$

In lemma 4.7, we established that there exists such a bijection, so we need to show naturality. First, however, we consider what the functors' actions on arrows are. Given an arrow $f : \mathcal{L} \to \mathcal{N}$, we have that the arrow $(- \times \mathcal{M})(f)$ must have as domain $\mathcal{L} \times \mathcal{M}$ and as codomain $\mathcal{N} \times \mathcal{M}$. Therefore, we set $(- \times \mathcal{M})(f) = f \times \mathrm{id}$. The arrow $(\mathcal{M} \to -)(f)$ has as domain $\mathcal{M} \to \mathcal{L}$ and as codomain $\mathcal{M} \to \mathcal{N}$, so $(\mathcal{M} \to -)(f) = f \circ -$.

To show naturality, let two arrows $f : \mathcal{L} \to \mathcal{L}'$ and $g : \mathcal{N}' \to \mathcal{N}$ be given. We are to show that the diagram

$$\begin{tikzcd}
CLat(\mathcal{L} \times \mathcal{M}, \mathcal{N}) \arrow[r, "m_{\mathcal{L},\mathcal{N}}"] & CLat(\mathcal{L}, (\mathcal{M} \to \mathcal{N})) \\
CLat(\mathcal{L}' \times \mathcal{M}, \mathcal{N}') \arrow[u, "CLat((-\times\mathcal{M})(f),g)"] \arrow[r, "m_{\mathcal{L}',\mathcal{N}'}"'] & CLat(\mathcal{L}', (\mathcal{M} \to \mathcal{N}')) \arrow[u, "CLat(f,(\mathcal{M}\to-)(g))"']
\end{tikzcd}$$

commutes. So, let an $\alpha : \mathcal{L}' \times \mathcal{M} \to \mathcal{N}$ be given. The "up–right" direction of the diagram yields $m_{\mathcal{L},\mathcal{N}}$ taken on the composite

$$\mathcal{L} \times \mathcal{M} \xrightarrow{f\times\mathrm{id}} \mathcal{L}' \times \mathcal{M} \xrightarrow{\alpha} \mathcal{N}' \xrightarrow{g} \mathcal{N}$$

This gives the function $m_{\mathcal{L},\mathcal{N}}(g \circ \alpha \circ f \times \mathrm{id}) : \mathcal{L} \to (\mathcal{M} \to \mathcal{N})$ given by

$$(m_{\mathcal{L},\mathcal{N}}(g \circ \alpha \circ f \times \mathrm{id})(x))(y) = (g \circ \alpha \circ f \times \mathrm{id})(x, y) = g(\alpha(f(x), y))$$

From the "right–up" direction, we get the composite

$$\mathcal{L} \xrightarrow{f} \mathcal{L}' \xrightarrow{m_{\mathcal{L}',\mathcal{N}'}(\alpha)} (\mathcal{M} \to \mathcal{N}') \xrightarrow{g\circ-} (\mathcal{M} \to \mathcal{N})$$

This is the function that given an $x \in \mathcal{L}$ returns the function in $(\mathcal{M} \to \mathcal{N})$ that given a $y \in \mathcal{M}$ produces the value

$$\begin{aligned}
&(((g \circ -) \circ m_{\mathcal{L}',\mathcal{N}'}(\alpha) \circ f)(x))(y) \\
&= g(((m_{\mathcal{L}',\mathcal{N}'}(\alpha) \circ f)(x))(y)) \\
&= g((m(\alpha)(f(x)))(y)) \\
&= g(\alpha(f(x), y))
\end{aligned}$$

This means that we have shown the naturality we needed, and the theorem is proved. □

Remark 4.9. The naturality of the bijection in the preceding proof can be obtained more quickly. One could just remark that the category SET of sets and functions between them is cartesian closed, and the natural bijection in this adjunction is exactly the same as the one we have used here. Therefore, the bijection in our proof is also natural.

This completes our work with complete lattice. In the next section we are going to deal with a subcategory of *CLat* that is closely related to the equilogical spaces.

5 Algebraic Lattices

Until now we have studied the category of complete lattices and continuous functions. In this section, we are going to deal with a full subcategory, namely that of the *algebraic lattices* (and continuous functions between them). We are going to define this category, study some of its properties, and show that this category is also cartesian closed. This will help us in showing the same property for the equilogical spaces. We will find that we have already done a great deal of the work in this regard.

An algebraic lattice is a complete lattice that satisfies an additional restriction. Before we can formulate this property, however, we need to pick out a certain kind of element in a complete lattice:

Definition 5.1. Let $\mathcal{L} = (L, \leq)$ be a complete lattice. A *compact* element e of $\mathcal{L}$ is an element with the property that if $e \leq \bigvee S$ for a subset $S \subseteq L$, then there exists a *finite* subset $S_0 \subseteq S$ such that $e \leq \bigvee S_0$. The set of compact elements in the lattice $\mathcal{L}$ is denoted $\mathcal{K}(\mathcal{L})$.

Example 5.2. We have seen that $\mathcal{P}(A)$ is a complete lattice for any set A. It is easy to see that the compact elements in this lattice are the finite subsets of A.

Notice that the bottom element $\perp_{\mathcal{L}}$ of a complete lattice is always compact, and that in a finite lattice, all elements are compact. We wish to show that the least upper bound of a finite set of compact elements is again compact, but for this we will first need an easy lemma. It is a consequence of lemma 2.10 in [DP].

Lemma 5.3. *Let P be a complete lattice, and assume that $S, T \subseteq P$. Then*

$$\bigvee(S \cup T) = (\bigvee S) \vee (\bigvee T) \quad \textit{and} \quad \bigwedge(S \cup T) = (\bigwedge S) \vee (\bigwedge T)$$

Proof: We just deal with the least upper bounds. Note that $\bigvee S, \bigvee T \leq (\bigvee S) \vee (\bigvee T)$, so $(\bigvee S) \vee (\bigvee T)$ is an upper bound for S and T and thus $S \cup T$. If a is another such upper bound, then it is an upper bound of both S and T, so $\bigvee S, \bigvee T \leq a$, in particular $(\bigvee S) \vee (\bigvee T) \leq a$. □

Corollary 5.4. *In a complete lattice, least upper bounds respect finite unions, that is: if* $S_0 = S_1 \cup \cdots \cup S_n$*, then*

$$\bigvee S_0 = \bigvee S_1 \vee \cdots \vee \bigvee S_n$$

Proof: Easy induction using lemma 5.3. □

This enables us to prove

Lemma 5.5. *Let* $k_1, \dots, k_n$ *be compact elements in a complete lattice* $(P, \leq)$*. Then the element* $k = k_1 \vee, \cdots \vee k_n$ *is also compact.*

Proof: We just show the claim for $n = 2$, the result then follows by easy induction. Let k_1, k_2 be compact elements, and set $k = k_1 \vee k_2$. We are to show that k is compact, so let $S \subseteq P$ be a set with $k \leq \bigvee S$. Then we also have that

$$k_1 \leq \bigvee S \quad \text{and} \quad k_2 \leq \bigvee S$$

This means that there exist finite sets $S_1, S_2 \subseteq S$ such that

$$k_1 \leq \bigvee S_1 \quad \text{and} \quad k_2 \leq \bigvee S_2$$

Let $S_0 = S_1 \cup S_2 \subseteq S$. Then we claim that $k \leq \bigvee S_0$. This will complete the proof of the lemma, since S_0 is finite. By corollary 5.4, it suffices to show that $k \leq \bigvee S_1 \vee \bigvee S_2$, i.e. $k_1 \vee k_2 \leq \bigvee S_1 \vee \bigvee S_2$. But this is immediate from the equations above. □

We now define the algebraic lattices. They are complete lattices where the compact elements are "characterizing":

Definition 5.6. A complete lattice $\mathcal{L} = (L, \leq)$ is called *algebraic* if every element is determined by the compact elements that it is greater than or equal to. In other words, the condition is that for any two elements, if the implication

$$e \leq x \Leftrightarrow e \leq y$$

holds for all compact elements $e \in \mathcal{K}(\mathcal{L})$, then $x = y$.

Example 5.7. The powerset lattice $\mathcal{P}(A)$ is algebraic since the singleton sets are compact.

Remark 5.8. If $\mathcal{L} = (L, \leq)$ is an algebraic lattice then we have for all elements $x \in L$ that

$$x = \bigvee \{e \in \mathcal{K}(\mathcal{L}) | e \leq x\}$$

To see this, note that x is indeed an upper bound for the set on the right hand side, so if a compact element e satisfies $e \leq \bigvee \{e \in \mathcal{K}(\mathcal{L}) | e \leq x\}$, then $e \leq x$. On the other hand, if $e \leq x$ for a compact e, then $e \in \{e \in \mathcal{K}(\mathcal{L}) | e \leq x\}$, and then $e \leq \bigvee \{e \in \mathcal{K}(\mathcal{L}) | e \leq x\}$. Since $\mathcal{L}$ is algebraic, this implies that $x = \bigvee \{e \in \mathcal{K}(\mathcal{L}) | e \leq x\}$.

Just like we defined the category *CLat*, we define the category of algebraic lattices:

Definition 5.9. The category *ALat* of algebraic lattices is the category with

- Algebraic lattices as objects
- Functions that are continuous with regard to the Σ–topology as morphisms

Note that since continuity of functions is the same thing as in the complete lattices, *ALat* is a full subcategory of *CLat*. It is the goal of this section to prove that this category is also cartesian closed. For this, we need to show that *ALat* is closed under the construction of function spaces. [DS98] lists two ways of doing this, and carries out one of them. We will carry out the other, both because it gives a better understanding and because the argument is more direct and intuitive.

For two algebraic lattices $\mathcal{L} = (L, \leq_{\mathcal{L}})$ and $\mathcal{M} = (M, \leq_{\mathcal{M}})$, we certainly have that the function space $(\mathcal{L} \to \mathcal{M})$ is a complete lattice, since in particular, $\mathcal{L}$ and $\mathcal{M}$ are complete. In order to show that it is indeed algebraic, we will characterize the compact elements in $(\mathcal{L} \to \mathcal{M})$ and show that there are "enough" of them to distinguish all elements of the space. The argument is based on so–called "step functions", and I have been inspired to consider these by functions in [DSn] of the same name (and same behaviour) between powersets. Let us first define these.

Definition 5.10. Let $\mathcal{L} = (L, \leq_{\mathcal{L}})$ and $\mathcal{M} = (M, \leq_{\mathcal{M}})$ be algebraic lattices and let $x_0 \in L, y_0 \in M$ be compact elements. We then define the *step function* $[x_0, y_0] : L \to M$ by

$$[x_0, y_0](x) = \text{if } x_0 \leq_{\mathcal{L}} x \text{ then } y_0 \text{ else } \bot_{\mathcal{M}}$$

Let us first show that we have in fact defined an element of $(\mathcal{L} \to \mathcal{M})$:

Lemma 5.11. *Let $[x_0, y_0] : L \to M$ be a step function between algebraic lattices as in definition 5.10. Then $[x_0, y_0]$ is continuous.*

Proof: As always, we show that lubs of directed sets are preserved, so let $D \subseteq L$ be a directed set. Then there either is an element $d \in D$ such that $x_0 \leq_{\mathcal{L}} d$, or all $d \in D$ satisfy $d \not\leq_{\mathcal{L}} x_0$. In the first case we have $x_0 \leq_{\mathcal{L}} \bigvee_{\mathcal{L}} D$, and therefore $[x_0, y_0](\bigvee_{\mathcal{L}} D) = y_0$. At the same time, $\bigvee_{\mathcal{M}} [x_0, y_0](D) = y_0$. In the other case, we have that $\bigvee_{\mathcal{M}} [x_0, y_0](D) = \bigvee_{\mathcal{M}} \{\bot_{\mathcal{M}}\} = \bot_{\mathcal{M}}$, so we need to show that $[x_0, y_0](\bigvee_{\mathcal{L}} D) = \bot_{\mathcal{M}}$. Suppose therefore that $x_0 \leq \bigvee_{\mathcal{L}} D$. Since x_0 is compact, there exists a finite $D_0 \subseteq D$ such that $x_0 \leq_{\mathcal{L}} \bigvee_{\mathcal{L}} D_0$. But since D is directed, $\bigvee_{\mathcal{L}} D_0 \in D$, so we have obtained a contradiction. Therefore, $x_0 \not\leq_{\mathcal{L}} \bigvee_{\mathcal{L}} D$, and we have the desired equality. □

What makes the step functions useful in our quest is the fact that they are "generic" in a sense that we shall see later, and that they are in fact compact:

Lemma 5.12. *Let $\mathcal{L} = (L, \leq_{\mathcal{L}})$ and $\mathcal{M} = (M, \leq_{\mathcal{M}})$ be algebraic lattices and let $x_0 \in L, y_0 \in M$ be compact elements. Then $[x_0, y_0] : L \to M$ is a compact element of $(\mathcal{L} \to \mathcal{M})$.*

Proof: Let $F \subseteq (\mathcal{L} \to \mathcal{M})$ be a set such that

$$[x_0, y_0] \leq_{(\mathcal{L} \to \mathcal{M})} \bigvee_{(\mathcal{L} \to \mathcal{M})} F$$

This means that for all $x \in L$, we have that $[x_0, y_0](x) \leq_{\mathcal{M}} \bigvee_{\mathcal{M}} \{f(x) | f \in F\}$, in particular $y_0 = [x_0, y_0](x_0) \leq_{\mathcal{M}} \bigvee_{\mathcal{M}} \{f(x_0) | f \in F\}$. Since y_0 is compact, there exists a finite subset $F_0 \subseteq F$ such that $y_0 \leq_{\mathcal{M}} \bigvee_{\mathcal{M}} \{f(x_0) | f \in F_0\}$. We claim that $[x_0, y_0] \leq_{(\mathcal{L} \to \mathcal{M})} F_0$, and note that this is enough to prove the lemma. So let an $x \in L$ be given. If $x_0 \leq_{\mathcal{L}} x$, then

$$[x_0, y_0](x) = y_0 \leq_{\mathcal{M}} \bigvee_{\mathcal{M}} \{f(x_0) | f \in F_0\} \leq_{\mathcal{M}} \bigvee_{\mathcal{M}} \{f(x) | f \in F_0\} = (\bigvee_{(\mathcal{L} \to \mathcal{M})} F_0)(x)$$

If $x_0 \not\leq_{\mathcal{L}} x$, we get that

$$[x_0, y_0](x) = \perp_{\mathcal{M}} \leq_{\mathcal{M}} (\bigvee_{(\mathcal{L} \to \mathcal{M})} F_0)(x)$$

In both cases, we have the desired inequality, so $[x_0, y_0] \leq_{(\mathcal{L} \to \mathcal{M})} F_0$, and $[x_0, y_0]$ is a compact element. ☐

Earlier, it was claimed that the step functions are in a certain sense generic. Here comes the theorem that explains that statement:

Theorem 5.13. *For algebraic lattices $\mathcal{L}$ and $\mathcal{M}$, all elements in $(\mathcal{L} \to \mathcal{M})$ is the least upper bound of a set of step functions $[x_0, y_0]$ with $x_0 \in \mathcal{K}(\mathcal{L}), y_0 \in \mathcal{K}(\mathcal{M})$.*

Proof: Let $f \in (\mathcal{L} \to \mathcal{M})$, and define

$$g = \bigvee_{(\mathcal{L} \to \mathcal{M})} \{[x_0, y_0] | x_0 \in \mathcal{K}(\mathcal{L}), y_0 \in \mathcal{K}(\mathcal{M}), y_0 \leq_{\mathcal{M}} f(x_0)\}$$

We then have

$$\begin{array}{rcll} g(x) & = & \bigvee_{\mathcal{M}} \{[x_0, y_0](x) | x_0 \in \mathcal{K}(\mathcal{L}), y_0 \in \mathcal{K}(\mathcal{M}), y_0 \leq_{\mathcal{M}} f(x_0)\} & \\ & = & \bigvee_{\mathcal{M}} \{y_0 \in \mathcal{K}(\mathcal{M}) | \exists x_0 \in \mathcal{K}(\mathcal{L}) : (x_0 \leq_{\mathcal{L}} x \ \& \ y_0 \leq_{\mathcal{M}} f(x_0))\} & (1) \\ & = & \bigvee_{\mathcal{M}} \{f(x_0) | x_0 \in \mathcal{K}(\mathcal{L}), x_0 \leq_{\mathcal{L}} x\} & (2) \\ & = & \bigvee_{\mathcal{M}} f(D), \text{ where } D = \{x_0 \in \mathcal{K}(\mathcal{L}) | x_0 \leq_{\mathcal{L}} x\} & \\ & = & f(\bigvee_{\mathcal{L}} D) & (3) \\ & = & f(x) & (4) \end{array}$$

We explain the numbered equations here:

1. Follows from the definition of $[x_0, y_0]$ and the fact that $\bot_{\mathcal{M}}$ is a compact element.
2. Holds because $f(x_0) = \bigvee_{\mathcal{M}}\{y \in \mathcal{K}(\mathcal{L}) | y \leq_{\mathcal{M}} f(x_0)\}$.
3. The set D is directed, since least upper bounds of finite sets of compact elements are again compact, so since f is continuous, the equation holds.
4. This equation holds since x is the least upper bound of the compact elements it is greater than or equal to.

This means that $f = \bigvee_{(\mathcal{L}\to\mathcal{M})}\{[x_0, y_0] | x_0 \in \mathcal{K}(\mathcal{L}), y_0 \in \mathcal{K}(\mathcal{M}), y_0 \leq_{\mathcal{M}} f(x_0)\}$, so the theorem is proved. □

This theorem leads to a characterization of the compact elements in $(\mathcal{L} \to \mathcal{M})$.

Corollary 5.14. *The compact elements in the function space* $(\mathcal{L} \to \mathcal{M})$ *between two algebraic lattices are precisely the least upper bounds of finite sets of step functions.*

PROOF: Since every step function is compact, the least upper bound of a finite set of step functions is also compact by lemma 5.5. If f is a compact element of $(\mathcal{L} \to \mathcal{M})$, then we have that f is equal to the least upper bound of a set F of step functions by theorem 5.13. Since f is compact, there exists a finite subset $F_0 \subseteq F$ of step functions such that $f \leq_{(\mathcal{L}\to\mathcal{M})} \bigvee_{(\mathcal{L}\to\mathcal{M})} F_0$. Since $\bigvee_{(\mathcal{L}\to\mathcal{M})} F_0 \leq_{(\mathcal{L}\to\mathcal{M})} \bigvee_{(\mathcal{L}\to\mathcal{M})} F$, this must mean that $f = \bigvee_{(\mathcal{L}\to\mathcal{M})} F_0$, as desired. □

With this knowledge about the compact elements of $(\mathcal{L} \to \mathcal{M})$ at hand, it is not hard to show that *ALat* is closed under the construction of function spaces.

Theorem 5.15. *Given two algebraic lattices* $\mathcal{L} = (L, \leq_{\mathcal{L}}), \mathcal{M} = (M, \leq_{\mathcal{M}})$*, the function space* $(\mathcal{L} \to \mathcal{M})$ *is again an algebraic lattice.*

PROOF: We know that $(\mathcal{L} \to \mathcal{M})$ is complete, so we need to show the requirement for algebraicness. Therefore, let two distinct elements $g, h \in (\mathcal{L} \to \mathcal{M})$ be given. We then need to show that the statement

$$\forall f \in \mathcal{K}(\mathcal{L} \to \mathcal{M}) : f \leq_{(\mathcal{L}\to\mathcal{M})} g \Leftrightarrow f \leq_{(\mathcal{L}\to\mathcal{M})} h$$

does not hold. Since g and h are distinct, there is an element $x' \in L$ such that $g(x') \neq h(x')$. There is also a compact element $x_0 \in L$ with $g(x_0) \neq h(x_0)$, for if there were not, we could use the fact that $\{x_0 \in \mathcal{K}(\mathcal{L}) | x_0 \leq_{\mathcal{L}} x'\} \subseteq L$ is directed to conclude that

$$\begin{array}{rcl}
g(x') & = & g(\bigvee_{\mathcal{L}}\{x_0 \in \mathcal{K}(\mathcal{L})|x_0 \leq_{\mathcal{L}} x'\}) \\
& = & \bigvee_{\mathcal{M}} g(\{x_0 \in \mathcal{K}(\mathcal{L})|x_0 \leq_{\mathcal{L}} x'\}) \\
& = & \bigvee_{\mathcal{M}} h(\{x_0 \in \mathcal{K}(\mathcal{L})|x_0 \leq_{\mathcal{L}} x'\}) \\
& = & h(\bigvee_{\mathcal{L}}\{x_0 \in \mathcal{K}(\mathcal{L})|x_0 \leq_{\mathcal{L}} x'\}) \\
& = & h(x')
\end{array}$$

which is a contradiction. Now there are three possibilities: either (i): $g(x_0) < h(x_0)$, (ii): $h(x_0) < g(x_0)$, or (iii): $g(x_0)$ and $h(x_0)$ are incomparable. In case (i), we must have a compact element $y_0 \in M$ such that $y_0 \leq_{\mathcal{M}} h(x_0)$, but $y_0 \not\leq_{\mathcal{M}} g(x_0)$, since $\mathcal{M}$ is an algebraic lattice. Then, since h is monotone, we have that $[x_0, y_0] \leq_{(\mathcal{L}\to\mathcal{M})} h$, but $[x_0, y_0](x_0) = y_0 \not\leq_{\mathcal{M}} g(x_0)$, so $[x_0, y_0] \not\leq_{(\mathcal{L}\to\mathcal{M})} g$. Case ($ii$) is dual. For case ($iii$), we note that there must exist a compact element $y_0 \in M$ with *either* $y_0 \leq_{\mathcal{M}} h(x_0)$ and $y_0 \not\leq_{\mathcal{M}} g(x_0)$ *or* $y_0 \leq_{\mathcal{M}} g(x_0)$ and $y_0 \not\leq_{\mathcal{M}} h(x_0)$, since otherwise we could conclude that $g(x_0) = h(x_0)$ because $\mathcal{M}$ is algebraic. In either of the cases, proceed as in case (i). The theorem is proved. ☐

This theorem does a great deal of the job of proving that *ALat* is indeed cartesisan closed. It is easy to see that in a product lattice $\mathcal{L} \times \mathcal{M}$ of two complete lattices, an element (x, y) is compact if and only if both x and y are compact in $\mathcal{L}$ and $\mathcal{M}$, respectively. This also means that $\mathcal{L} \times \mathcal{M}$ is algebraic iff $\mathcal{L}$ and $\mathcal{M}$ are. Moreover, *ALat* has as terminal object the algebraic lattice $\{\star\}$ and we can conclude that *ALat* has all finite products. Note also that since continuity between two algebraic lattices is the same as between complete ones, we have that lemma 4.7 applies to algebraic lattices. Finally, the proof of 4.8 extends to *ALat*, so we can conclude

Theorem 5.16. *The category* ALat *of algebraic lattices and continuous functions between them is a cartesian closed category.*

Before we end this section, we take a look at the Σ–topology for an algebraic lattice. It turns out that it has a basis that we will need later.

Theorem 5.17. *Let $\mathcal{L} = (L, \leq)$ be a complete lattice. For a compact element $e \in \mathcal{L}$, define the set $D_e \subseteq L$ by*

$$D_e = \{x \in L | e \leq x\}$$

Then $\{D_e | e \in \mathcal{K}(\mathcal{L})\}$ is a basis for the topology $\Sigma_{\mathcal{L}}$.

Proof: First, we must establish that the sets $D_e, e \in \mathcal{K}(L)$ are open. Their upwards closure is obvious, so we show that D_e satisfies the condition (C) for any compact $e \in L$. Therefore, let a set $S \subseteq L$ have $\bigvee S \in D_e$, that is $e \leq \bigvee S$. By compactness of e, there exists a finite $S_0 \subseteq S$ with $e \leq \bigvee S_0$. But then $\bigvee S_0 \in D_e$, so we have shown that D_e is open. To show that $\{D_e | e \in \mathcal{K}(\mathcal{L})\}$ is indeed a basis, let an open set W be given. It then suffices to show that

$$W = \bigcup_{e \in W \cap \mathcal{K}(\mathcal{L})} D_e,$$

To prove this, let a compact $e \in W$ be given, and assume that $x \in D_e$. Then $e \leq x$, but this means that $x \in W$, since W is upwards closed. This means that $\bigcup_{e\in W} D_e \subseteq W$. For the reverse inclusion, let an $x \in W$ be given. By remark 5.8, we get that $x = \bigvee\{e \in \mathcal{K}(\mathcal{L})|e \leq x\} \in W$. Since W is open, and in particular satisfies the condition (C), there exist compact $e_1 \leq x, \dots, e_n \leq x$ such that $e_1 \vee \cdots \vee e_n \in W$. Let $e = e_1 \vee \cdots \vee e_n$. Then, by defintion, $e \leq x$, and by lemma 5.5, e is compact. This means that

$$x \in D_e \subseteq \bigcup_{e\in W} D_e,$$

and the claim is proved. □

5.1 The E Theorems

In this section, we are going to prove two theorems, the Embedding and the Extension Theorems, that are going to come in hand when we treat the equilogical spaces. They deal more with the field of topology than our previous work.

Recall that a topological embedding of one topological space into another is a function that is continuous, open, and injective. The first theorem in this section says that any T_0–space can be embedded into a powerset (which is an algebraic lattice, as we have seen).

Theorem 5.18 (The Embedding Theorem). *Let $\mathcal{X} = (X, \mathcal{T})$ be a T_0–space. Then the mapping $x \mapsto \mathcal{T}(x)$ is a topological embedding of $\mathcal{X}$ into $\mathcal{P}(\mathcal{T})$ considered as a space with the Σ–topology.*

Proof: We are to show that $x \mapsto \mathcal{T}(x)$ is injective, continuous, and open. Since $\mathcal{X}$ is a T_0-space, the mapping is clearly an injection. We know that $\mathcal{P}(\mathcal{T})$ is an algebraic lattice, so a basis for the Σ–topology can be deduced from theorem 5.17 and example 5.2 and is given by

$$\{\mathcal{M} \in \mathcal{P}(\mathcal{T})|\mathcal{U} \subseteq \mathcal{M}\} = \{\mathcal{M} \subseteq \mathcal{T}|\mathcal{U} \subseteq \mathcal{M}\},$$

where $\mathcal{U} = \{U_1, \dots, U_n\}$ ranges over the finite subsets of $\mathcal{T}$. This means that any open set in the space $(\mathcal{P}(\mathcal{T}), \Sigma)$ is a union of such sets. To show that $x \mapsto \mathcal{T}(x)$ is continuous, it thus suffices to show that the inverse image of such a set is open. The inverse image of $\{\mathcal{M} \subseteq \mathcal{T}|\mathcal{U} \subseteq \mathcal{M}\}$ is the set

$$\begin{aligned} \{x \in X|\mathcal{T}(x) \in \{\mathcal{M} \subseteq \mathcal{T}|\mathcal{U} \subseteq \mathcal{M}\}\} &= \{x \in X|\mathcal{U} \subseteq \mathcal{T}(x)\} \\ &= \{x \in X|x \in U \text{ for all } U \in \mathcal{U}\} \\ &= U_1 \cap \cdots \cap U_n \end{aligned}$$

This last set is obviously open. This means that $x \mapsto \mathcal{T}(x)$ is continuous. As a last step, we must show that the image of an open set U in $\mathcal{X}$ is open in $\mathcal{P}(\mathcal{T})$. The image of U under the mapping is

$$\{\mathcal{T}(x)|x \in U\} = \bigcup_{x \in U} \{\mathcal{M} \in \mathcal{P}(\mathcal{T})|\{U\} \subseteq \mathcal{M}\}$$

This is a union of basis sets and it is therefore open. This means that our mapping is open, and then the proof of the theorem is complete. □

The other theorem in this subsection shows that algebraic lattices have a nice property inasmuch as continuous functions into them can be extended. After the proof of the theorem, an informal discussion of the related notions in other texts will follow.

Theorem 5.19 (The Extension Theorem). *Let $\mathcal{L} = (L, \leq)$ be an algebraic lattice, and let $\mathcal{X} = (X, \mathcal{T}_{\mathcal{X}})$ and $\mathcal{Y} = (Y, \mathcal{T}_{\mathcal{Y}})$ be topological spaces such that $\mathcal{X}$ is a subspace of $\mathcal{Y}$. If the function $f : X \to L$ is continuous (with respect to the Σ–topology on L), then f has a continuous extension to all of Y.*

Proof: Assume that the function $f : X \to L$ is continuous. We then define a function $\overline{f} : Y \to L$ by

$$\begin{array}{rcl} \overline{f}(y) & = & \bigvee\{(\bigwedge\{f(x)|x \in U \cap X\})|U \in \mathcal{T}_{\mathcal{Y}}(y)\} \\ & = & \bigvee\{(\bigwedge\{f(x)|x \in U \cap X\})|y \in U\} \end{array}$$

We claim that this is the desired extension, so we must show that $\overline{f}$ extends f and is continuous.

For the first claim, let $x \in X$. Then, for any open $U \in \mathcal{T}_{\mathcal{X}}$, we have that $\bigwedge\{f(x)|x \in U \cap X\} \leq f(x)$, so by definition of $\overline{f}$, we have that $\overline{f}(x) \leq f(x)$. Since $\mathcal{L}$ is algebraic, it now suffices to show that for any compact element $e \in L$, we have that $e \leq f(x)$ implies $e \leq \overline{f}(x)$. Therefore, let $e \leq f(x)$ for a compact element e. Recall from theorem 5.17 that the set $D_e = \{x \in L|e \leq x\}$ is open. Since f is continuous, we have that $f^{-1}(D_e)$ is open in $\mathcal{X}$. Also note that if $x' \in f^{-1}(D_e)$, then $f(x') \in D_e$, so $e \leq f(x')$. This means that $e \leq \bigwedge\{f(x')|x' \in X \cap f^{-1}(D_e)\}$, so $e \leq \overline{f}(x)$, and $\overline{f}$ extends f.

To show that $\overline{f}$ is continuous, we show that the inverse image of a basic open set D_e is open. Let a $y \in Y$ be such that $\overline{f}(y) \in D_e$ for a compact element $e \in L$. This means that

$$\bigvee\{\bigwedge\{f(x)|x \in U \cap X\}|y \in U\} \in D_e$$

Since D_e satisfies the condition (C), we can find $U_1, \ldots, U_n \in \mathcal{T}_{\mathcal{Y}}(y)$ such that

$$\bigvee_{i=1}^{n}\{\bigwedge\{f(x)|x \in U_i \cap X\}\} \in D_e \quad \text{i.e. that}$$

$$e \leq \bigvee_{i=1}^{n}\{\bigwedge\{f(x)|x \in U_i \cap X\}\}$$

Let the open set $U_0 \in \mathcal{T}_\mathcal{Y}$ be defined as $U_0 = U_1 \cap \cdots \cap U_n$, and note that $y \in U_0$. Since $U_0 \subseteq U_i$ for $i \in \{1, \ldots, n\}$, we have that

$$\bigwedge\{f(x)|x \in U_i \cap X\} \leq \bigwedge\{f(x)|x \in U_0 \cap X\}$$

This means that

$$\begin{aligned} e &\leq \bigvee_{i=1}^{n}\{\bigwedge\{f(x)|x \in U_i \cap X\}\} \\ &\leq \bigvee_{i=1}^{n}\{\bigwedge\{f(x)|x \in U_0 \cap X\}\} \\ &= \bigwedge\{f(x)|x \in U_0 \cap X\} \end{aligned}$$

Now, if $y' \in U_0$, then we have that $U_0 \in \mathcal{T}_\mathcal{Y}(y')$, so $e \leq \overline{f}(y')$, i.e. $\overline{f}(y') \in D_e$, so $y' \in \overline{f}^{-1}(D_e)$. We have thus shown that for any $y \in \overline{f}^{-1}(D_e)$, there is an open set $U_0(y)$ containing y and such that $U_0(y) \subseteq \overline{f}^{-1}(D_e)$. This means that

$$\overline{f}^{-1}(D_e) = \bigcup_{y \in \overline{f}^{-1}(D_e)} U_0(y),$$

so $\overline{f}^{-1}(D_e)$ is open in $\mathcal{Y}$, as desired. $\square$

In the terminology of [GG] and [DS72], we have shown that any algebraic lattice is an *injective space* (see definition 1.1 in [DS72]). To show this quickly with references from these texts, we note that any algebraic lattice is a *continuous lattice* by theorem I.4.5 in [GG], and then we can conclude by theorem 2.11 in [DS72] that it is injective. To avoid too many new terms and a lot of technicalities, however, we have rewritten the proof of 2.11 in [DS72] so that it shows the theorem directly for algebraic lattices.

6 Fixed Point Theorems

This section can be viewed as a digression inasmuch as the results herein are not neccesary for the development of the rest of the paper. However, an introduction to lattices and order would be incomplete without these results, and they also illustrate further understanding of concepts in [DSn] and [DP], and all the results herein are stated as one single, unproved theorem in [DS98]. We will show Knaster–Tarski's theorem about fixed points for monotone mappings on complete lattices, Tarski's theorem regarding the set of fixed points for a monotone self–function on a complete lattice, Kleene's theorem on the least fixed point of a continuous function, and Davis' theorem which is a reverse to the Knaster–Tarski theorem. The proof of the last result is rather technical, and we will not go into all details.

6.1 Knaster–Tarski's Theorem

Knaster–Tarski's Theorem says that any monotone self–function on a complete lattice has a least fixed point.

Theorem 6.1 (Knaster–Tarski, 1929). *Let $\mathcal{L} = (L, \leq)$ be a complete lattice, and let $f : L \to L$ be monotone. Then f has a least fixed point.*

Proof: Define the set $H \subseteq L$ of *prefixed* points by

$$H = \{x \in L | f(x) \leq x\}$$

Note that $f(\top) \leq \top$, so $H \neq \varnothing$. Let $\alpha = \bigwedge H$. Then, for all $x \in H$, $\alpha \leq x$ and since f is monotone, $f(\alpha) \leq f(x) \leq x$. This means that $f(\alpha)$ is a lower bound for H, and by definition of α, we get $f(\alpha) \leq \alpha$. Moreover, $f(f(\alpha)) \leq f(\alpha)$, so $f(\alpha) \in H$. But then $\alpha \leq f(\alpha)$, and we can conclude that $\alpha = f(\alpha)$, so α is a fixed point. Note that H contains all fixed points, so α is a lower bound for all fixed points. This means that α is the least fixed point. □

By dual reasoning, one can also show that f also has a greatest fixed point $\beta = \bigvee\{x \in L | x \leq f(x)\}$.

Example 6.2. Consider the mapping $\varphi : \mathbb{N} \to \mathbb{N}$ given by $\varphi(n) = n + 1$. The theorem above says that this mapping has a fixed point, since φ is clearly monotone. Intuitively, this doesn't sound right, but of course the problem is that $\mathbb{N}$ is not a complete lattice unless we equip it with a top element ∞, which is the desired fixed point.

6.2 Tarski's Theorem

Tarski's theorem says that in a complete lattice, the fixed points of a monotone self-function also constitute a complete lattice. Note that by the previous theorem, this set has a top and a bottom element.

Theorem 6.3 (Tarski, 1939). *Let $\mathcal{L} = (L, \leq)$ be a complete lattice, and let $f : L \to L$ be monotone. Then the set of fixed points of f with the inherited ordering is also a complete lattice.*

Proof: Let $\mathcal{FP}(f)$ be the set of fixed points which we are to show is a complete lattice. Therefore, let a set $S \subseteq \mathcal{FP}(f)$ be given. Since also $S \subseteq L$, we have a least upper bound $\bigvee S \in L$. Define a new function $g : L \to L$ by

$$g(x) = f(x) \vee (\bigvee S).$$

Clearly, it is monotone, so it has a least fixed point $p \in L$. This means that

$$g(p) = \bigvee S \vee f(p) = p,$$

so $\bigvee S \leq p$. This means that $x \leq p$ for all $x \in S$. We wish to show that $p \in \mathcal{FP}(f)$, since this means that p is an upper bound of S in $\mathcal{FP}(f)$. Since f is monotone, we have that $x = f(x) \leq f(p)$ for all $x \in S$, i.e. $f(p)$ is an upper bound for S in L. But then $\bigvee S \leq f(p)$, and since $\bigvee S \vee f(p) = p$, this implies $p = f(p)$, so $p \in \mathcal{FP}(f)$. If $y \in \mathcal{FP}(f)$ is another upper bound for S in $\mathcal{FP}(f)$, then we have $\bigvee S \leq y$, so $y = \bigvee S \vee y = \bigvee S \vee f(y) = g(y)$. This means that y is a fixed point of g, but by definition this means that $p \leq y$, so p is indeed the least upper bound for S in $\mathcal{FP}(f)$. $\square$

6.3 Kleene's Theorem

Kleene's Theorem gives an explicit formula for the least fixed point of a self–function on a complete lattice in case the function is *continuous*.

Theorem 6.4. *Assume that $\mathcal{L} = (L, \leq)$ is a complete lattice, and let $f : \mathcal{L} \to \mathcal{L}$ be continuous. Then the least fixed point of f is given by*

$$\bigvee_{n=0}^{\infty} f^n(\bot)$$

Proof: By corollary 3.14, f is monotone, so f has a least fixed point. Define the set $D \subseteq L$ by

$$D = \{f^n(\bot) | n \in \mathbb{N}\}$$

We claim that D is directed. This is easily proved, since an easy induction argument shows that $f^n(\bot) \leq f^{n+1}(\bot)$ for all $n \in \mathbb{N}$, and then each finite subset of D has as its least upper bound "the element with the highest n". Now, since f is continuous, we have

$$f(\bigvee_{n=0}^{\infty} f^n(\bot)) = f(\bigvee \{f^n(\bot) | n \in \mathbb{N}\}) = f(\bigvee D)$$

$$= \bigvee f(D) = \bigvee f(\{f^n(\bot) | n \in \mathbb{N}\}) = \bigvee_{n=0}^{\infty} f^{n+1}(\bot)$$

Now, $\bigvee_{n=0}^{\infty} f^{n+1}(\bot) = \bigvee_{n=1}^{\infty} f^n(\bot) = \bigvee_{n=0}^{\infty} f^n(\bot)$, since we just add $\bot$ on the last equation, and this does not change a least upper bound. This means that $\bigvee_{n=0}^{\infty} f^n(\bot)$ is a fixed point. If a is any other fixed point, then $f^n(a) = a$ for all $n \in \mathbb{N}$, so $\bigvee_{n=0}^{\infty} f^n(a) = a$. Since the function $x \mapsto \bigvee_{n=0}^{\infty} f^n(x)$ is monotone and $\bot \leq a$, we get that $\bigvee_{n=0}^{\infty} f^n(\bot) \leq \bigvee_{n=0}^{\infty} f^n(a)$, as desired. $\square$

6.4 Davis' Theorem

Anne Davis' theorem from 1950 is a converse to Knaster–Tarski's theorem, inasmuch as it shows the converse of that result. It is stated in [DP] as theorem 4.17 and it is "difficult to prove". Indeed, we will not go into all details here, but refer to the original source for the proof of the following

Lemma 6.5. *Let $\mathcal{L} = (L, \leq)$ be a lattice. If $\mathcal{L}$ is not complete, there exist two nets $(b_i)_{i<\alpha}$ and $(c_j)_{j<\beta}$ in L where α and β are limit ordinals such that*

- *(i)* $b_i < c_j$ *for all* $i < \alpha, j < \beta$.
- *(ii)* $(b_i)_{i<\alpha}$ *is strictly increasing and* $(c_j)_{j<\beta}$ *is strictly decreasing.*
- *(iii)* *There is no* $l \in L$ *such that* l *is an upper bound of* $(b_i)_{i<\alpha}$ *and a lower bound for* $(c_j)_{j<\beta}$.

Proof: See [AD]. The proof relies heavily on the Axiom of Choice, and it is an exercise in dealing with ordinals.

With this result, we can show the reverse of Knaster–Tarski's theorem:

Theorem 6.6 (Davis, 1950). *Let $\mathcal{L} = (L, \leq)$ be a lattice such that every monotone function $f : L \to L$ has a fixed point. Then $\mathcal{L}$ is a complete lattice.*

Proof: The proof is by contradiction. Assume that $\mathcal{L}$ is not complete and that every monotone self–function has a fixed point. Let two nets $(b_i)_{i<\alpha}$ and $(c_j)_{j<\beta}$ fulfill the conditions $(i) - (iii)$ from lemma 6.5. We aim at defining a monotone function $f : L \to L$ without any fixed points. Let $x \in L$ be given. We define $f(x)$ by cases. If x is not an upper bound of $(b_i)_{i<\alpha}$, then $\{i < \alpha | b_i \not\leq x\}$ is not empty, and there exists a least $i_0 < \alpha$ such that $b_{i_0} \not\leq x$. Set $f(x) = b_{i_0}$ in this case, and note that $f(x) \neq x$. If x is an upper bound of $(b_i)_{i<\alpha}$, then x is not a lower bound of $(c_j)_{j<\beta}$ because of requirement (ii) in 6.5. This means that $\{j < \beta | x \not\leq c_j\} \neq \varnothing$, and there exists a smallest $j_0 < \beta$ such that $x \not\leq c_{j_0}$. Set $f(x) = c_{j_0}$ here, and note that this definition implies that f has no fixed points. If we can show that it is monotone, we will have finished the proof of the theorem. Therefore, let $x \leq y$ in L. Again we argue by cases.

(1) If y is not an upper bound for $(b_i)_{i<\alpha}$, then x is not either, and we will have

$$I_x = \{i < \alpha | b_i \not\leq x\} \supseteq \{i < \alpha | b_i \not\leq y\} = I_y \neq \varnothing$$

Therefore, the smallest element i_x in I_x is smaller than or equal to that of I_y (call it i_y), and since $(b_i)_{i<\alpha}$ is increasing,

$$f(x) = b_{i_x} \leq b_{i_y} = f(y)$$

(2) If y is an upper bound for $(b_i)_{i<\alpha}$, but x is not, then $f(x) \in (b_i)_{i<\alpha}$ and $f(y) \in (c_j)_{j<\beta}$, so $f(x) \leq f(y)$ by condition (ii) in the lemma.

(3) If both y and x are upper bounds for $(b_i)_{i<\alpha}$, then we have that

$$\varnothing \neq J_x = \{j < \beta | x \nleq c_j\} \subseteq \{j < \beta | y \nleq c_j\} = J_y$$

Thus, the smallest element $j_0 \in J_y$ is smaller than or equal to the smallest element j_x in J_x. Since $(c_j)_{j<\beta}$ is decreasing, this means that

$$f(y) = c_{j_y} \geq c_{j_x} = f(x)$$

This takes care of all possibilities, hence f is monotone. $\square$

7 Some Category Theory

In this section, we will show a result that characterizes equivalences of categories. This result is handy in many settings, and we will use it later to show that the equilogical spaces are equivalent to a category that relates to *ALat*. Before we embark on the theorem, let us recall the relevant definitions:

Definition 7.1.

- A natural transformation $\mu : F \Rightarrow G$ is called a *natural isomorphism* if all its components are isomorphisms.
- A functor $F : \mathcal{D} \to \mathcal{C}$ is called an *equivalence of categories* if there is a functor $G : \mathcal{C} \to \mathcal{D}$ and natural isomorphisms $\mu : FG \Rightarrow \mathrm{id}_{\mathcal{C}}, \nu : GF \Rightarrow \mathrm{id}_{\mathcal{D}}$.
- An *adjoint equivalence* between two categories $\mathcal{C}$ and $\mathcal{D}$ is an adjuction $(G, F, \eta, \varepsilon) :\ \mathcal{C} \underset{G}{\overset{F}{\leftrightarrows}} \mathcal{D}$ where both the unit $\eta : \mathrm{id}_{\mathcal{D}} \Rightarrow GF$ and the co–unit $\varepsilon : FG \Rightarrow \mathrm{id}_{\mathcal{C}}$ are natural isomorphisms.
- A functor $G : \mathcal{C} \to \mathcal{D}$ is called *essentially surjective on objects* if for any object $D \in \mathcal{D}$, there is an object $C \in \mathcal{C}$ such that $G(C)$ is isomorphic to D.

Note that if $(G, F, \eta, \varepsilon) :\ \mathcal{C} \underset{G}{\overset{F}{\leftrightarrows}} \mathcal{D}$ is an adjoint equivalence, then we have that $\eta^{-1} = \{(\eta_D)^{-1} | D \in \mathcal{D}\} : GF \Rightarrow \mathrm{id}_{\mathcal{D}}$ and $\varepsilon^{-1} = \{(\varepsilon_C)^{-1} | C \in \mathcal{C}\} : \mathrm{id}_{\mathcal{C}} \Rightarrow FG$ are natural isomorphisms (naturality is easy to check). The adjunction $(G, F, \eta, \varepsilon)$ implies that we have the *triangular identities*

$$\begin{tikzcd}
G \arrow[r, Rightarrow, "\eta\star G"] \arrow[dr, Rightarrow, "\mathrm{id}_G"'] & GFG \arrow[d, Rightarrow, "G\diamond\varepsilon"] & F \arrow[r, Rightarrow, "F\diamond\eta"] \arrow[dr, Rightarrow, "id_F"'] & FGF \arrow[d, Rightarrow, "\varepsilon\star F"] & , \\
& G & & F &
\end{tikzcd}$$

where $\eta \star G$ is $\{\eta_{GC} | C \in \mathcal{C}\}$ and $G \diamond \varepsilon$ means $\{G(\varepsilon_C) | C \in \mathcal{C}\}$. Note that the composition $\eta \star G \cdot \eta^{-1} \star G$ has as its components $\eta_{GC} \circ \eta_{GC}^{-1} = \mathrm{id}_{GC}$, so it is the identity transformation id : $G \Rightarrow G$. Similarly for the $\diamond$–operator. This means that we have the identities

$$\begin{aligned} &G \diamond \varepsilon \cdot \eta \star G \cdot \eta^{-1} \star G \cdot G \diamond \varepsilon^{-1} = \mathrm{id}_G \quad \text{and} \\ &\varepsilon \star F \cdot F \diamond \eta \cdot F \diamond \eta^{-1} \cdot \varepsilon^{-1} \star F = \mathrm{id}_F, \end{aligned}$$

so we can write a new set of identities of transformations:

$$\begin{tikzcd} G \arrow[r, Rightarrow, "G\diamond\varepsilon^{-1}"] \arrow[dr, Rightarrow, "\mathrm{id}_G"'] & GFG \arrow[d, Rightarrow, "\eta^{-1}\star G"] \\ & G \end{tikzcd} \qquad \begin{tikzcd} F \arrow[r, Rightarrow, "\varepsilon^{-1}\star F"] \arrow[dr, Rightarrow, "\mathrm{id}_F"'] & FGF \arrow[d, Rightarrow, "F\diamond\eta^{-1}"] \\ & F \end{tikzcd}$$

By exercise 86 in [Jaap] (or thm. IV.2 in [ML]), this means that $(F, G, \varepsilon^{-1}, \eta^{-1})$ is also an adjunction (and in fact an adjoint equivalence), and in particular, G is both a left and a right adjoint to F.

The following theorem relate the notions from definition 7.1. It is taken from [ML], page 93.

Theorem 7.2. *Let $F : \mathcal{D} \to \mathcal{C}$ be a functor. Then the following statements are equivalent:*

(*i*) *F is part of an adjoint equivalence: $(G, F, \eta, \varepsilon)$:* $\begin{tikzcd} \mathcal{C} \arrow[r, shift right, "G"'] & \mathcal{D} \arrow[l, shift right, "F"'] \end{tikzcd}$.

(*ii*) *F is an equivalence of categories.*

(*iii*) *F is full, faithful, and essentially surjective on objects.*

Proof: (*i*) $\Rightarrow$ (*ii*): This is obvious from the definitions. Choose the right adjoint G as the opposite functor, and the given natural isomorphisms.

(*ii*) $\Rightarrow$ (*iii*): Assume that $F : \mathcal{D} \to \mathcal{C}$ is an equivalence of categories. We first show that F is essentially surjective on objects. Therefore, let an object $C \in \mathcal{C}$ be given. We have a functor $G : \mathcal{C} \to \mathcal{D}$ and a natural isomorphism $\mu : FG \Rightarrow \mathrm{id}_{\mathcal{C}}$, so the following diagram commutes:

$$\begin{tikzcd} FG(C) \arrow[r, "\mu_C"] \arrow[d, "\mathrm{id}_{FG(C)}"'] & C \arrow[d, "\mathrm{id}_C"] \\ FG(C) \arrow[r, "\mu_C"'] & C \end{tikzcd}$$

Since μ_C is an isomorphism, $C \simeq F(GC)$, and as $GC \in D$, this means that F is essentially surjective. Next we show that F is faithful, so let $f : D \to D'$ be an arrow. We have a natural isomorphism $\nu : GF \Rightarrow \mathrm{id}_{\mathcal{D}}$, so we have the following commutative diagram

$$\begin{tikzcd}
GF(D) \arrow[r, "\nu_D"] \arrow[d, "GF(f)"'] & D \arrow[d, "f"] \\
GF(D') \arrow[r, "\nu_{D'}"'] & D'
\end{tikzcd}$$

Here, both ν_D and $\nu_{D'}$ are iso, so we have that

$$f = \nu_{D'} \circ GF(f) \circ \nu_D^{-1}$$

Now, if $F(f) = F(g) : F(D) \to F(D')$, then

$$f = \nu_{D'} \circ GF(f) \circ \nu_D^{-1} = \nu_{D'} \circ GF(g) \circ \nu_D^{-1} = g$$

This means that F is faithful. By using the natural isomorphism $\mu : FG \Rightarrow \mathrm{id}_{\mathcal{C}}$ again, an analogous argument shows that G is also faithful. It remains to show that F is full, so let a morphism $h : F(D) \to F(D')$ in $\mathcal{C}$ be given. Consider the arrow $G(h) : GF(D) \to GF(D')$ in $\mathcal{D}$, and the isomorphisms ν_D and $\nu_{D'}$ from before, and define $\varphi = \nu_{D'} \circ G(h) \circ \nu_D^{-1} : D \to D'$. We then have the following commutative diagrams:

$$\begin{tikzcd}
GFD \arrow[r, "\nu_D"] \arrow[d, "G(h)"'] & D \arrow[d, "\varphi"] \\
GFD' \arrow[r, "\nu_{D'}"'] & D'
\end{tikzcd}
\qquad
\begin{tikzcd}
GFD \arrow[r, "\nu_D"] \arrow[d, "GF(\varphi)"'] & D \arrow[d, "\varphi"] \\
GFD' \arrow[r, "\nu_{D'}"'] & D'
\end{tikzcd}$$

We then see that $G(h) = GF(\varphi)$, and since G is faithful, this means that $h = F(\varphi)$, i.e. F is full, as desired.

(iii) $\Rightarrow$ *(i)*: Suppose $F : \mathcal{D} \to \mathcal{C}$ is full, faithful and essentially surjective on objects. We first construct a left adjoint functor $G : C \to D$. Let an object $C \in \mathcal{C}$ be given. Since F is essentially surjective on objects, there is an object $D_0 \in \mathcal{D}$ (call this object $G_0(C)$), and an isomorphism $\eta_C : C \simeq FG_0(C)$. Now, given an object $D \in \mathcal{D}$ and an arrow $f : C \to F(D)$ in $\mathcal{C}$:

$$\begin{tikzcd}
C \arrow[r, "\eta_C"] \arrow[dr, "f"'] & FG_0C \arrow[d, dotted, "F(g)"] \\
& F(D)
\end{tikzcd}
\qquad g : G_0C \to D$$

We have that $f \circ \eta_C^{-1} : FG_0C \to FD$ has the form $F(g)$ for some $g : G_0C \to D$ since F is full. Since F is faithful, this g is unique, and we have shown that the arrow η_C is universal. By theorem IV.2.*ii* in [ML] (or a dual of exercise 88 in [Jaap]), we can extend G_0 to a left adjoint G of F,[1] and the unit of the adjunction

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[1] Note that we use the Axiom of Choice to define $G_0(C) = D_0$ above.

becomes the natural isomorphism $\eta = \{\eta_C | C \in \mathcal{C}\} : \mathrm{id}_{\mathcal{C}} \Rightarrow FG$ (note that G is the *left* adjoint unlike the examples in [Jaap]). All we need to show is that the counit $\varepsilon : GF \Rightarrow \mathrm{id}_{\mathcal{D}}$ for this adjunction has isomorphisms at all its components. Therefore, let an object $D \in \mathcal{D}$ be given. By the triangular identites for our adjunction, we have that $F\varepsilon_D \circ \eta_{FD} = \mathrm{id}_{FD}$, so $F\varepsilon_D$ is invertible. Since F is full, $(F\varepsilon_D)^{-1} = \eta_{FD} : FD \to FGFD$ has the form $F(k)$ for a $k : D \to GFD$ in $\mathcal{D}$. We then have $F(\varepsilon_D \circ k) = F\varepsilon_D \circ \eta_{FD} = \mathrm{id}_{FD}$. Since F is faithful, this means that $\varepsilon_D \circ k = \mathrm{id}_D$. Similarly it is seen that k is also a left inverse, so ε_D is iso. This completes the proof of the theorem. □

The following corollaries tell us that equivalent categories share many properties.

Corollary 7.3. *If the functor $F : \mathcal{D} \to \mathcal{C}$ is an equivalence of categories, then F preserves all limits and colimits which exist in $\mathcal{D}$.*

Proof: By the preceding theorem, there exists a functor G that is both a left and a right adjoint to F. The statement now follows from theorem 5.2 in [Jaap]. □

In particular, if one of two eqivalent categories has finite products, then so does the other. The next result in this section says that if two categories are equivalent, and one of them is cartesian closed, then so is the other.

Theorem 7.4. *Assume that the functor $F : \mathcal{D} \to \mathcal{C}$ is an equivalence of categories and that $\mathcal{D}$ is a cartesian closed category. Then $\mathcal{C}$ is also cartesian closed.*

Proof: We know that $\mathcal{D}$ is a cartesian closed category, so it has finite products; by the preceding results we conclude that $\mathcal{C}$ has finite products too. What we need to show is that for some object $X \in \mathcal{C}$, the product functor $- \times X : \mathcal{C} \to \mathcal{C}$ has a right adjoint. This amounts to show that for three objects $X, Y, Z \in \mathcal{C}$, we can find an exponential object W such that there is a natural bijection

$$\frac{X \times Y \to Z}{\overline{X \to W}}$$

between the hom–sets $\mathcal{C}(X \times Y, Z)$ and $\mathcal{C}(X, W)$. We first construct a bijection in several steps, and afterwards, we will explain why each of the steps are natural. Let $G : \mathcal{C} \to \mathcal{D}$ be the left and right adjoint that exists by theorem 7.2, and let ε and η be the unit and counit of the adjunction $F \dashv G$.

As a first step we show that there is a bijection $m_{X \times Y, Z}$:

$$\frac{X \times Y \to Z}{\overline{G(X \times Y) \to G(Z)}}$$

Given an arrow $f : X \times Y \to Z$, we let $m_{X \times Y, Z}(f) = G(f) : G(X \times Y) \to G(Z)$. Since G is full and faithful, $m_{X \times Y, Z}$ is a bijection. Now note that G is an equivalence of categories, so it preserves products, and $G(X \times Y) = G(X) \times G(Y)$. Therefore we have a bijection

$$\begin{array}{c} X \times Y \to Z \\ \hline\hline G(X) \times G(Y) \to G(Z) \end{array}$$

The second step in the construction of our bijection is this:

$$\begin{array}{c} G(X) \times G(Y) \to G(Z) \\ \hline\hline G(X) \to G(Z)^{G(Y)} \end{array}$$

The existence of this bijection follows from the assumption that $\mathcal{D}$ is a cartesian closed category. The third step is like the first: since F is full and faithful, there is a bijection

$$\begin{array}{c} G(X) \to G(Z)^{G(Y)} \\ \hline\hline FG(X) \to F(G(Z)^{G(Y)}) \end{array}$$

The fourth and last step is to construct a bijection

$$\begin{array}{c} FG(X) \to F(G(Z)^{G(Y)}) \\ \hline\hline X \to F(G(Z)^{G(Y)}) \end{array}$$

Before we do this, let us first set $W = F(G(Z)^{G(Y)})$. Recall that we have a natural isomorphism $\varepsilon : FG \Rightarrow \mathrm{id}_C$. Given a morphism $f : FG(X) \to W$, we set $m_{FGX,W}(f) = f \circ \varepsilon_X^{-1} : X \to W$. Then, for an arrow $g : X \to W$, we set $m_{FGX,W}^{-1}(g) = g \circ \varepsilon_X$. It is obvious that these two operations are mutually inverse. All in all, we have now constructed an object W and shown that there is a bijection:

$$\begin{array}{c} X \times Y \to Z \\ \hline\hline X \to W \end{array}$$

What is left to show is that each of the steps in this bijection are natural. The naturality of the last step follows from naturality of ε, and the second step's naturality follows from the fact that $\mathcal{D}$ is a cartesian closed category. The third and the first step's naturality both emerge from the same reason, so we just argue for the first one. Showing the desired naturality comes down to showing that for two morphisms $f : X \to X'$ and $g : Z' \to Z$ in $\mathcal{C}$, the following diagram commutes (We have used that G preserves products):

$$\begin{tikzcd}
\mathcal{C}(X \times Y, Z) \arrow[r, "m_{X,Z}"] & \mathcal{D}(G(X) \times G(Y), G(Z)) \\
\mathcal{C}(X' \times Y, Z') \arrow[u, "\mathcal{C}(f\times\mathrm{id},g)"] \arrow[r, "m_{X',Z'}"'] & \mathcal{D}(G(X') \times G(Y), G(Z')) \arrow[u, "\mathcal{D}(G(f)\times\mathrm{id},G(g))"']
\end{tikzcd}$$

Therefore, let a morphism $h : X' \times Y \to Z'$ in $\mathcal{C}$ be given. Applying $\mathcal{C}(f \times \mathrm{id}, g)$ to it yields the following morphism:

$$X \times Y \xrightarrow{f\times\mathrm{id}} X' \times Y \xrightarrow{h} Z' \xrightarrow{g} Z$$

The bijection $m_{X,Z}$ just applies the functor G, so the up–right direction yields the morphism $G(g) \circ G(h) \circ G(f) \times G(\mathrm{id})$. From the other direction we get $\mathcal{D}(G(f) \times \mathrm{id}, G(g))$ applied to $G(h)$. This is the morphism

$$G(X) \times G(Y) \xrightarrow{G(f)\times\mathrm{id}} G(X') \times G(Y) \xrightarrow{G(h)} G(Z') \xrightarrow{G(g)} G(Z)$$

We see that these two are indeed the same morphism, and the proof is complete. □

We have now seen that two equivalent share the properies of having finite limits and cartesian closure; that is: if one of two equivalent categories have one of these properties, then the other category also has this property. In fact, this is true for almost every categorical property, and in particular for the properties that we are working with in this paper. It will be too tedious to derive all this, so we just formulate it as an unproved fact:

Thesis 7.5. *Assume that $\mathcal{C}$ and $\mathcal{D}$ are two equivalent categories. If $\mathcal{C}$ has a categorical property (e.g.: "is complete", "has all limits", "is cartesian closed", etc.), then $\mathcal{D}$ also has this property.*

We now have all the tools we need to explore the basics of the equilogical spaces at hand, so let's not wait any longer:

8 Equilogical Spaces and the Partial Counterparts

As we saw in section 2, the property of cartesian closure is not one shared by *TOP*, the category of topological spaces. The equilogical spaces is a solution to this problem inasmuch as it in a sense expands the category *TOP* to a "larger" category *EQU* that naturally "contains" *TOP*.

8.1 Equilogical Spaces

Here is the definition of the equilogical spaces. After the definition, we will show that they do indeed form a category.

Definition 8.1. The category *EQU* of *equilogical spaces* is defined as the category with

- Structures $\mathcal{E} = (E, \mathcal{T}_\mathcal{E}, \equiv_\mathcal{E})$ as objects. Here, $(E, \mathcal{T}_\mathcal{E})$ is a T_0–space, and $\equiv_\mathcal{E}$ is an equivalence relation on the set E. These structures are called equilogical spaces.
- Morphisms between two equilogical spaces $\mathcal{E} = (E, \mathcal{T}_\mathcal{E}, \equiv_\mathcal{E})$ and $\mathcal{F} = (F, \mathcal{T}_\mathcal{F}, \equiv_\mathcal{F}$) are equivalence classes modulo $\equiv_{\mathcal{E}\to\mathcal{F}}$ of continuous functions (w.r.t. $\mathcal{T}_\mathcal{E}$ and $\mathcal{T}_\mathcal{F}$) that respect the equivalence relations. The equivalence relation $\equiv_{\mathcal{E}\to\mathcal{F}}$ on mappings is defined by

  $$f \equiv_{\mathcal{E}\to\mathcal{F}} g \iff \forall x, y \in E : x \equiv_\mathcal{E} y \Rightarrow f(x) \equiv_\mathcal{F} g(y)$$

  The continuous maps that respect the equivalence relations are called *equivariant.*

What we need to show for this to make sense are the following:

1. For two equilogical spaces $\mathcal{E}$ and $\mathcal{F}$, the relation $\equiv_{\mathcal{E}\to\mathcal{F}}$ is an equivalence relation on the set of equivariant mappings.
2. *EQU* is a category.

We start out with the first claim, so let $\mathcal{E} = (E, \mathcal{T}_\mathcal{E}, \equiv_\mathcal{E})$ and $\mathcal{F} = (F, \mathcal{T}_\mathcal{F}, \equiv_\mathcal{F})$ be two equilogical spaces. We first show that $\equiv_{\mathcal{E}\to\mathcal{F}}$ is reflexive, so we let an equivariant map $f : E \to F$ be given. Since f is equivariant, $f(x) \equiv_\mathcal{F} f(y)$ for any $x, y \in E$ with $x \equiv_\mathcal{E} y$. This means that $\equiv_{\mathcal{E}\to\mathcal{F}}$ is indeed reflexive. To show that it is symmetric, assume that we have $f \equiv_{\mathcal{E}\to\mathcal{F}} g$, and let $x \equiv_\mathcal{E} y$. Then $f(x) \equiv_\mathcal{F} g(y)$, and $g(x) \equiv_\mathcal{F} f(y)$ follows easily by the fact that both f and g respect the equivalence relation. To show transitivity, let $f, g, h : E \to F$ be equivariant mappings with $f \equiv_{\mathcal{E}\to\mathcal{F}} g$ and $g \equiv_{\mathcal{E}\to\mathcal{F}} h$, and let $x \equiv_\mathcal{E} y$; we are to show that $f(x) \equiv_\mathcal{F} h(y)$. But by assumption, $f(x) \equiv_\mathcal{F} g(y) \equiv_\mathcal{F} g(x) \equiv_\mathcal{F} h(y)$, so the claim follows.

For the second statement, it suffices to show that we can calculate with representatives, since the identity maps are continuous and respect equivalence relations that are total; thus are representatives for the identity morphisms. This amounts to show that composition respect the equivalence relation on the function space. Therefore, assume that $f, f', g, g' : E \to F$ are equivariant mappings between the equilogical spaces $\mathcal{E}$ and $\mathcal{F}$ such that $f \equiv_{\mathcal{E}\to\mathcal{F}} f'$ and $g \equiv_{\mathcal{E}\to\mathcal{F}} g'$. We are then to show that $f \circ g \equiv_{\mathcal{E}\to\mathcal{F}} f' \circ g'$. We just show that $f \circ g \equiv_{\mathcal{E}\to\mathcal{F}} f \circ g'$; composition from the left can be handled similarly. So, let $x \equiv_\mathcal{E} y$. Then $g(x) \equiv_\mathcal{F} g'(y)$, and since f respects the equivalence relation, we can conclude that

$$(f \circ g)(x) = f(g(x)) \equiv_\mathcal{F} f(g'(y)) = (f \circ g')(y)$$

This shows that $f \circ g \equiv_{\mathcal{E} \to \mathcal{F}} f \circ g'$, and the statement follows. From now on, our calculations will deal only with representatives.

Remark 8.2. Note that there is a natural embedding of TOP into EQU. Just use equality as the equivalence relation for a topological space $(M, \mathcal{T})$.

8.2 Partial Equilogical Spaces

In order to show that the category EQU of equilogical spaces indeed has the cartesian closure that we desire, we introduce another category which, as it turns out, is closely related to EQU. Before the defintion, recall that a *partial* equivalence relation $\equiv$ on a set A is a relation on A that is symmetric and transitive (and only reflexive on some (possibly empty) subset of A). If for elements $x, y \in A$, we have that $x \equiv y$, then it follows by symmetry and transitivity of $\equiv$ that $x \equiv x$ and $y \equiv y$. Therefore, $\equiv$ splits A up into "partial equivalence classes", which resemble equivalence classes, but there may be elements that are not in any of these classes. We are going to use this later on. Here is the definition of the related category:

Definition 8.3. The category $PEQU$ of *partial equilogical spaces* is the category with

- Structures $\mathcal{A} = (A, \mathcal{T}_{\mathcal{A}}, \equiv_{\mathcal{A}})$ as objects. Here, we must have that $(A, \mathcal{T}_{\mathcal{A}})$ is the Σ–topology of an algebraic lattice and that $\equiv_{\mathcal{A}}$ is a partial equivalence relation on A.
- Morphisms between partial equilogical spaces $\mathcal{A}$ and $\mathcal{B}$ are equivalence classes modulo $\equiv_{\mathcal{A} \to \mathcal{B}}$ of continuous maps that respect the partial equivalence relations. The relation $\equiv_{\mathcal{A} \to \mathcal{B}}$ is defined just like in the case of the equilogical spaces.

Again, we have to show that $\equiv_{\mathcal{A} \to \mathcal{B}}$ is an equivalence relation and that $PEQU$ is indeed a category. These proofs are the same as the corresponding proofs in EQU, so we omit them.

We are going to show that $PEQU$ is cartesian closed, and afterwards, we will show that EQU and $PEQU$ are equivalent, thus obtaining the cartesian closure property for EQU. First, we need to show that $PEQU$ has finite products. It has as terminal object the partial equilogical space $(\{\star\}, \Sigma_{\{\star\}}, \{(\star, \star)\})$, and it has binary products since we can just take the structure with the cartesian product of the sets, the product topology, and the product relation.[2] In order

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[2] It is not hard to show that the product of two partial equivalence relations is again a partial equivalence relation.

to show cartesian closure of our category, we need to define function spaces in a suitable way. The way we will do this emerges from our previous knowledge about algebraic lattices. So, let $\mathcal{A} = (A, \Sigma_{\mathcal{A}}, \equiv_{\mathcal{A}})$ and $\mathcal{B} = (B, \Sigma_{\mathcal{B}}, \equiv_{\mathcal{B}})$ be partial equilogical spaces. Since $(A, \Sigma_{\mathcal{A}})$ and $(B, \Sigma_{\mathcal{B}})$ can be identified with algebraic lattices, we have previously worked out that the function space of all continuous functions is again an algebraic lattice; we will (naturally enough) choose this to be the "base" of our function space $(\mathcal{A} \to \mathcal{B})$; what we need to define is a partial equivalence relation on this space. We have already a candidate, namely $\equiv_{\mathcal{A}\to\mathcal{B}}$:

$$f \equiv_{\mathcal{A}\to\mathcal{B}} g \iff \forall x, y \in A : x \equiv_{\mathcal{A}} y \Rightarrow f(x) \equiv_{\mathcal{B}} g(y)$$

We know that this is a partial equivalence relation, so we have indeed defined our function space as an object of *PEQU*. What remains to be seen is that this definition of the function space implies that *PEQU* has the desired property of cartesian closure.

Theorem 8.4. *The category* PEQU *of partial equilogical spaces is a cartesian closed category.*

PROOF: This proof extends the proof of theorem 4.8. Referring to that proof, it suffices to show that for three partial equilogical spaces $\mathcal{A} = (A, \mathcal{T}_{\mathcal{A}}, \equiv_{\mathcal{A}})$, $\mathcal{B} = (B, \mathcal{T}_{\mathcal{B}}, \equiv_{\mathcal{B}})$, and $\mathcal{C} = (C, \mathcal{T}_{\mathcal{C}}, \equiv_{\mathcal{C}})$ there is a one–to–one correspondence between the two spaces

$$((\mathcal{A} \times \mathcal{B}) \to \mathcal{C}) \quad \text{and} \quad \mathcal{A} \to (\mathcal{B} \to \mathcal{C})$$

We have already established that there is a bijection between these spaces if we disregard the partial equivalence relation that correspond to the spaces. What we need to show is that this bijection is a bijection of equivalence classes and that it preserves the partial equivalence relations. We start with the second claim, and we must show that our bijection preserves the property of being equivariant; that is: if a map $\varphi : A \times B \to C$ preserves the partial equivalence relations, then (and only then) the map $m_{\mathcal{A},\mathcal{C}}(\varphi) = \Phi : A \to (B \to C)$, as defined in lemma 4.7, preserves the partial equivalence relations.

Therefore, assume that the map $\varphi \in \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ preserves the partial equivalence relation, that is, if $(y_1, y_1) \equiv_{\mathcal{A}\times\mathcal{B}} (x_2, y_2)$ (i.e. $x_1 \equiv_{\mathcal{A}} x_2$ and $y_1 \equiv_{\mathcal{B}} y_2$), then $\varphi(x_1, y_1) \equiv_{\mathcal{C}} \varphi(x_2, y_2)$. We must show that the map $\Phi : A \to (B \to C)$ defined by $x \mapsto \varphi_x = \varphi(x, -) : B \to C$ also preserves the partial equivalence relations on the spaces $\mathcal{A}$ and $(\mathcal{B} \to \mathcal{C})$. Therefore, let $x_1 \equiv_{\mathcal{A}} x_2$ in A. We must then show that $\varphi_{x_1} \equiv_{\mathcal{B}\to\mathcal{C}} \varphi_{x_2}$. We thus let $y_1 \equiv_{\mathcal{B}} y_2$ in B, and we are to show that $\varphi_{x_1}(y_1) \equiv_{\mathcal{C}} \varphi_{x_2}(y_2)$ in C. But since $\varphi_{x_i}(y_i) = \varphi(x_i, y_i), i = 1, 2$, this follows from our hypothesis.

For the reverse direction, assume that the map $\Phi : A \to (B \to C)$ preserves the partial equivalence relations. This means that if $x_1 \equiv_{\mathcal{A}} x_2$ in A, then $\Phi(x_1) \equiv_{\mathcal{B}\to\mathcal{C}}$

$\Phi(x_2)$, and this means that if $y_1 \equiv_{\mathcal{B}} y_2$ in B, then $(\Phi(x_1))(y_1) \equiv_{\mathcal{C}} (\Phi(x_2))(y_2)$. We must then show that the function $\varphi : A \times B \to C$ defined by $\varphi(x, y) = (\Phi(x))(y)$ respects the partial equivalence relations $\equiv_{\mathcal{A}\times\mathcal{B}}$ and $\equiv_{\mathcal{C}}$. So suppose $(x_1, y_1) \equiv_{\mathcal{A}\times\mathcal{B}} (x_2, y_2)$. We must then show that $\varphi(x_1, y_1) \equiv_{\mathcal{C}} \varphi(x_2, y_2)$. Again, this follows from our assumptions, since $\varphi(x_i, y_i) = (\Phi(x_i))(y_i)$ for $i \in \{1, 2\}$.

Finally, we have to show that m is indeed a bijection of equivalence classes, that is, if $\varphi_1 \equiv_{\mathcal{A}\times\mathcal{B}\to\mathcal{C}} \varphi_2$, then $\Phi_1 \equiv_{\mathcal{A}\to(\mathcal{B}\to\mathcal{C})} \Phi_2$, and vice versa. This requires an argument completely analogous to the one above. This completes the proof of the theorem. □

8.3 The Equivalence

In this section we are going to show a result that we promised in section 8.2, namely that the categories *EQU* and *PEQU* are equivalent. For this, we are going to use the work that we did in section 5.1 and 7. Here is the theorem:

Theorem 8.5. *The categories* EQU *and* PEQU *are equivalent.*

PROOF: In the light of theorem 7.2, it suffices to exhibit a functor $R : PEQU \to EQU$ that is full, faithful, and essentially surjective on objects. First a little notation: for a set A endowed with a partial equivalence relation $\equiv$, we define the set A_{tot} to be the set $\{x \in A | x \equiv x\} \subseteq A$. It is clear that $\equiv$ is reflexive (and an equivalence relation) when restricted to this subset. For a partial equilogical space $\mathcal{A} = (A, \mathcal{T}_{\mathcal{A}}, \equiv_{\mathcal{A}})$, we define

$$R(\mathcal{A}) = (A_{tot}, \mathcal{T}_{A_{tot}}, \equiv_{A_{tot}})$$

where $\mathcal{T}_{A_{tot}}$ and $\equiv_{A_{tot}}$ are the inherited topology, and the restriction of $\equiv_{\mathcal{A}}$ (which is an equivalence relation), respectively. We define the image of a equivariant map (this terminology is extended to *PEQU* in the obvious way) to be the restriction of the map to the subset where the equvalence relation is "total". First we need to show that this is in fact a functor. Note that $\mathcal{T}_{A_{tot}}$ separates points in A_{tot} since the Σ–topology of an algebraic lattice does so and the restriction of a T_0–space is again a T_0–space. Therefore, $R(\mathcal{A})$ is an equilogical space. Next, we need to show that for (a representative of) a map $f : \mathcal{A} = (A, \mathcal{T}_{\mathcal{A}}, \equiv_{\mathcal{A}}) \to \mathcal{B} = (B, \mathcal{T}_{\mathcal{B}}, \equiv_{\mathcal{B}})$ in *PEQU*, $R(f)$ is valid as (a representative of) a map $R(f) : R(\mathcal{A}) \to R(\mathcal{B})$ in *EQU*. The codomain of $R(f)$ is indeed B_{tot}, since if $x \equiv_{\mathcal{A}} x$, then $f(x) \equiv_{\mathcal{B}} f(x)$. It certainly respects the equivalence relation $\equiv_{A_{tot}}$, since f respects $\equiv_{\mathcal{A}}$, and we thus need to show that $R(f)$ is continuous, but the fact that restrictions of continuous functions are again continuous is well–known. It is clear that R preserves identities and compositions, and therefore, it is indeed a functor.

First we show that R is faithful, so let two partial equilogical spaces $\mathcal{A} = (A, \mathcal{T}_{\mathcal{A}}, \equiv_{\mathcal{A}})$ and $\mathcal{B} = (B, \mathcal{T}_{\mathcal{B}}, \equiv_{\mathcal{B}})$ be given. Showing that R is faithful amounts to showing that if to equivariant maps have images under R that are equivalent, then the two maps are equivalent. Therefore, assume that for two equivariant maps, $f, g : A \to B$, $R(f) \equiv_{R(\mathcal{A}) \to R(\mathcal{B})} R(g)$. We need to show that $f \equiv_{\mathcal{A} \to \mathcal{B}} g$, so let $x, y \in A$ satisfy $x \equiv_{\mathcal{A}} y$. This means that $x \equiv_{\mathcal{A}} x$ and $y \equiv_{\mathcal{A}} y$, and therefore

$$f(x) = (R(f))(x) \equiv_{B_{tot}} (R(g))(y) = g(y)$$

Since $\equiv_{B_{tot}}$ is a restriction of $\equiv_{\mathcal{B}}$, this means that $f(x) \equiv_{\mathcal{B}} g(y)$, so R is faithful.

Next, we show that R is full, and again we let two partial equilogical spaces $\mathcal{A} = (A, \mathcal{T}_{\mathcal{A}}, \equiv_{\mathcal{A}})$ and $\mathcal{B} = (B, \mathcal{T}_{\mathcal{B}}, \equiv_{\mathcal{B}})$ be given. Assume that $f : R(\mathcal{A}) \to R(\mathcal{B})$ is a morphism between the two images. We are to show that there is a valid map $\overline{f} : A \to B$ in $PEQU$ such that $R(\overline{f}) = f$. Note that f is a continuous map $f : (A_{tot}, \mathcal{T}_{A_{tot}}) \to (B, \mathcal{T}_{\mathcal{B}})$. Since $(B, \mathcal{T}_{\mathcal{B}})$ is the Σ-topology of an algebraic lattice, the Extension Theorem 5.19 tells us that f has a continuous extension $\overline{f} : A \to B$. First we need to show that this is a valid map in $PEQU$. We know that it is continuous, so we need to show that it preserves the partial equivalence relation. Therefore, we let $x, y \in A$ be such that $x \equiv_{\mathcal{A}} y$. Again, this means that $x, y \in A_{tot}$, so since f was a valid map in EQU and $\overline{f}$ extends f, $\overline{f}(x) \equiv_{\mathcal{B}} \overline{f}(b)$. We need to prove that $R(\overline{f}) = f$, but this follows easily since $\overline{f}$ was the extension of f from the set A_{tot}.

The last thing we need to prove is that R is essentially surjective. For this purpose, we let an equilogical space $\mathcal{E} = (E, \mathcal{T}_{\mathcal{E}}, \equiv_{\mathcal{E}})$ be given. We are then to construct a partial equilogical space $\mathcal{A}$ such that $R(\mathcal{A})$ is isomorphic to $\mathcal{E}$. For this we will use the Embedding Theorem 5.18. As the underlying set for $\mathcal{A}$, we take the set $\mathcal{P}(\mathcal{T}_{\mathcal{E}})$, and we naturally choose the Σ–topology as the topology on this algebraic lattice. What is left to construct is a partial equivalence relation $\equiv_{\mathcal{P}(\mathcal{T}_{\mathcal{E}})}$ such that we can use the Embedding Theorem to conclude $R(\mathcal{A}) \simeq \mathcal{E}$. For two elements $A, B \in \mathcal{P}(\mathcal{T}_{\mathcal{E}})$, we define $A \equiv_{\mathcal{P}(\mathcal{T}_{\mathcal{E}})} B$ as follows:

$$A \equiv_{\mathcal{P}(\mathcal{T}_{\mathcal{E}})} B \iff \exists x, y \in E : x \equiv_{\mathcal{E}} y \wedge A = \mathcal{T}_{\mathcal{E}}(x) \wedge B = \mathcal{T}_{\mathcal{E}}(y)$$

It is not difficult to see that this is a symmetric relation on $\mathcal{P}(\mathcal{T}_{\mathcal{E}})$. Assume that $A \equiv_{\mathcal{P}(\mathcal{T}_{\mathcal{E}})} B$ and $B \equiv_{\mathcal{P}(\mathcal{T}_{\mathcal{E}})} C$ for three elements $A, B, C \in \mathcal{P}(\mathcal{T}_{\mathcal{E}})$. This means that there exist $x, y, z, w \in E$ with

$$x \equiv_{\mathcal{E}} y, \quad A = \mathcal{T}_{\mathcal{E}}(x), \quad B = \mathcal{T}_{\mathcal{E}}(y), \quad z \equiv_{\mathcal{E}} w, \quad B = \mathcal{T}_{\mathcal{E}}(z), \quad \text{and} \quad C = \mathcal{T}_{\mathcal{E}}(w)$$

Note that since $(E, \mathcal{T}_{\mathcal{E}})$ is a T_0–space, the two conditions $B = \mathcal{T}_{\mathcal{E}}(y)$ and $B = \mathcal{T}_{\mathcal{E}}(z)$ imply that $y = z$. This means that

$$x \equiv_{\mathcal{E}} w \wedge A = \mathcal{T}_{\mathcal{E}}(x) \wedge C = \mathcal{T}_{\mathcal{E}}(w),$$

and we can conclude that $A \equiv_{\mathcal{P}(\mathcal{T}_{\mathcal{E}})} C$. This shows that $\equiv_{\mathcal{P}(\mathcal{T}_{\mathcal{E}})}$ is also a transitive relation. If we set $D = \mathcal{P}(\mathcal{T}_{\mathcal{E}})$ then it is easy to use reflexivity of $\equiv_{\mathcal{E}}$ and the

T_0-property of $\mathcal{T}_\mathcal{E}$ to see that a subset of $\mathcal{T}_\mathcal{E}$ is in D_{tot} if and only if it is the neighborhood filter of some point in E. Therefore, we get that the underlying set of $R(\mathcal{A}) \in EQU$ is $\{\mathcal{T}_\mathcal{E}(x) | x \in E\}$, and the topology $\mathcal{T}_{R(\mathcal{A})}$ and equivalence relation $\equiv_{R(\mathcal{A})}$ are the ones inherited from this. The Embedding Theorem says that the map $\varphi : (E, \mathcal{T}_\mathcal{E}) \to (\{\mathcal{T}_\mathcal{E}(x) | x \in E\}, \mathcal{T}_{R(\mathcal{A})})$ defined by $\varphi(x) = \mathcal{T}_\mathcal{E}(x)$ is an isomorphism. All that is left to show is that φ and $\varphi^{-1} : (\{\mathcal{T}_\mathcal{E}(x) | x \in E\}, \mathcal{T}_{R(\mathcal{A})}) \to (E, \mathcal{T}_\mathcal{E})$ given by $\varphi^{-1}(\mathcal{T}_\mathcal{E}(x)) = x$ respect the equivalence relations, so that they are valid as maps in EQU. If $x, y \in E$ are equivalent, then it is clear that $\varphi(x) = \mathcal{T}_\mathcal{E}(x) \equiv_{\mathcal{P}(\mathcal{T}_\mathcal{E})} \mathcal{T}_\mathcal{E}(y) = \varphi(y)$. Conversely, $\mathcal{T}_\mathcal{E}(x) \equiv_{\mathcal{P}(\mathcal{T}_\mathcal{E})} \mathcal{T}_\mathcal{E}(y)$ implies that $x \equiv_\mathcal{E} y$, so φ^{-1} does indeed respect the equivalence relations. This completes the proof of the theorem. □

As an easy consequence, we derive at our goal of this chapter:

Corollary 8.6. *The category* EQU *of equilogical spaces is a cartesian closed category.*

Proof: This follows immediately from theorems 8.4, 8.5, and 7.4. □

With reference to [Jaap, chp. 7], we now see that we can model the typed λ–calculus with our category. We shall see in the next section that EQU also has other properties.

9 Assemblies and Modest Sets

In the project description for this paper (see appendix 10), it is stated that that EQU can be used to model dependent type theory. Although it is indeed so, the subject is highly complicated, and for reasons of time and space, we will not get to that point in this paper. What we will do, however, is to relate to one of the subjects in [Jaap]. In this section we will thus show (along with other properties) that EQU is a *regular* category. This means that we can give a sound interpretation of the regular fragment of first–order logic, as it is done in [Jaap, chp. 4].

The way we will do this is to present yet another category that again is equivalent to EQU, and then we will work out the results in this category. With our thesis 7.5 in hand, we will conclude that EQU has these properties too.

Some technical remarks: More details are left to the reader in this section than in the previous sections. The reason for this is that a lot of these technicalities are straightforward to verify and that the project's main purpose, the cartesian closure of EQU, has already been established. In this section, we will write $(\mathcal{A}, \equiv_\mathcal{A})$ for an element of $PEQU$. It is then clear that the topology corresponding to this element is just the Σ–topology corresponding to $\mathcal{A}$. Also, when we say that $\mathcal{A}$ is an algebraic lattice, it will be understood that it has A and $\leq_\mathcal{A}$ as its underlying set and order, respectively (and similarly for other structures). This will be done in a way so that it cannot confuse.

We will define the category that is the subject of this section, and show that it is indeed equivalent to EQU. Our category is a full subcategory of $Assm(ALat)$, which is defined as follows:

Definition 9.1. The category $Assm(ALat)$ of *assemblies over the algebraic lattices* is defined as follows:

- *Objects* are triples $(X, \mathcal{A}, E_X)$, where X is a set, $\mathcal{A}$ is an algebraic lattice, and where $E_X : X \to \mathcal{P}(A)$ is a map such that $E_X(x) \neq \varnothing$ for all $x \in X$.
- A *morphism* between two structures $(X, \mathcal{A}, E_X)$ and $(Y, \mathcal{B}, E_Y)$ is a function $f : X \to Y$ such that there exists a continuous function $g : \mathcal{A} \to \mathcal{B}$ which satisfies:
$$\forall x \in X \forall a \in E_X(x) : g(a) \in E_Y(f(x))$$

We say that $E_X(x)$ is the set of *realizers* for x, and that the map g *tracks* f. We also use the realizer notion in connection with the maps, so we also say that g realizes f. Somtimes we will denote a morphism f in $Assm(ALat)$ by $\langle f, g\rangle$ to indicate that g tracks f. This should not be confused with a product morphism, and note that if both g_1 and g_2 tracks f, then $\langle f, g_1\rangle$ is the same morphism as $\langle f, g_2\rangle$.

To define the category that we will show is equivalent to EQU, we must pick out special objects of our category.

Definition 9.2. An object $(X, \mathcal{A}, E_X)$ is called *modest* if and only if

$$\forall x, x' \in X : x \neq x' \implies E_X(x) \cap E_X(x') = \varnothing$$

The full subcategory of $Assm(ALat)$ that results from only considering the modest objects is called the category of *modest sets over algebraic lattices* and is denoted $Mod(ALat)$. Note that in this category, a realizer $a \in E_X(x)$ determines the element x uniquely.

The reason we study this category in this setting comes from this theorem.

Theorem 9.3. *The categories* EQU*,* PEQU*, and* Mod(ALat) *are all equivalent.*

Proof: We just need to show that there exists a functor $F : Mod(ALat) \to PEQU$ that is full, faithfull, and essentially surjective. For an object $(X, \mathcal{A}, E_X)$, we define a partial equivalence relation $\equiv_{\mathcal{A}}$ on A by

$$a \equiv_{\mathcal{A}} a' \iff \exists x \in X : a, a' \in E_X(x)$$

(symmetry and transitivity are easy to check), and we set $F(X, \mathcal{A}, E_X) = (\mathcal{A}, \equiv_{\mathcal{A}})$. For a morphism $f : (X, \mathcal{A}, E_X) \to (Y, \mathcal{B}, E_Y)$ tracked by the continuous $g : A \to B$, we set $F(f) = [g]$. In order for this to make sense, we must show that g is indeed a equivariant map in *PEQU* and that different realizers for f result in maps that are equivalent (so that they have the same image). To show that realizer maps are equivariant, let $a \equiv_{\mathcal{A}} a'$ in the image space. Then there exists an $x \in X$ such that $a, a' \in E_X(x)$. Since g tracks f, $g(a), g(a') \in E_Y(f(x))$, but this means that $g(a) \equiv_{\mathcal{B}} g(a')$, so g is indeed equivariant. For the well-definedness of F, it remains to show that if g' is another realizer for f, then $g \equiv_{\mathcal{A}\to\mathcal{B}} g'$. So suppose this is the case and let $a \equiv_{\mathcal{A}} a'$ in the image space. Again, this means that there is an $x \in X$ such that $a, a' \in E_X(x)$. Since g, g' both track f, it is easy to deduce that $g(a) \equiv_{\mathcal{B}} g'(a')$, as desired. This means that $g \equiv_{\mathcal{A}\to\mathcal{B}} g'$, and that F is well–defined.

To show that F is full, consider an equivariant map $h : (\mathcal{A}, \equiv_{\mathcal{A}}) \to (\mathcal{B}, \equiv_{\mathcal{B}})$ between the images of $(X, \mathcal{A}, E_X)$ and $(Y, \mathcal{B}, E_Y)$. We must define a function $f : X \to Y$ such that h tracks f. That is, for an $x \in X$, we must define $f(x) \in Y$ such that for all $a \in E_X(x)$, we have that $h(x) \in E_Y(f(x))$. Note that $a \in E_X(x)$ means that $a \equiv_{\mathcal{A}} a$, and since h is equivariant, we have that $h(a) \equiv_{\mathcal{B}} h(a)$, i.e: there is a $y \in Y$ such that $h(a) \in E_Y(y)$. With the help of the Axiom of Choice, we can define $f(x) = y$, and this means that F is full.

To show that F is faithful, we let $f, f' : (X, \mathcal{A}, E_X) \to (Y, \mathcal{B}, E_Y)$ be two distinct maps in *Mod*(*ALat*), tracked by g, g', respectively. We must show that the images of these morphisms are distinct, so assume that $g \equiv_{\mathcal{A}\to\mathcal{B}} g'$ in the image, that is

$$\exists x \in X : a_1, a_2 \in E_X(x) \Longrightarrow \exists y \in Y : g(a_1), g'(a_2) \in E_Y(y)$$

Since $f \neq f'$, there exists an $x_0 \in X$ such that $f(x_0) \neq f'(x_0)$. Let $a_1, a_2 \in E_X(x_0)$. Then $g(a_1) \in E_Y(f(x_0))$, and since we are only dealing with modest objects, the implication above bespeaks that $g'(a_2) \in E_Y(f(x_0))$. But $g'(a_2) \in E_Y(f'(x_0))$, and this contradicts the modesty of $(Y, \mathcal{B}, E_Y)$.

Finally we show that F is essentially surjective. Let $(A, \equiv) \in PEQU$. As in the proof of 8.5, we construct the set $A_{tot} = \{a \in A | a \equiv a\}$ (on which $\equiv$ is an equivalence relation), and we set $X = A_{tot}/\equiv$. Also, we define $E_X : X \to \mathcal{P}(A)$ by $E([a]) = \{a' \in A | a' \equiv a\}$. We now claim that $F(X, \mathcal{A}, E_X) \simeq (\mathcal{A}, \equiv)$. All that needs to be shown is that $a_1 \equiv a_2 \iff a_1 \equiv_{\mathcal{A}} a_2$. This is an easy exercise. $\square$

With this result, we can use our thesis 7.5 to show that *EQU* possesses certain properties; we just have to show them in *Mod*(*ALat*). The goal will be to show that *Mod*(*ALat*) (and thereby *EQU*) is a regular category. Before we do this, let us verify fundamental properties of *Mod*(*ALat*). Some of these are already verified in *EQU*, and the derivations are therefore redundant, but it is a good illustration of how to work in *Mod*(*ALat*) to go through them.

Theorem 9.4. Mod(ALat) *(and thereby* EQU*) is a complete category.*

PROOF: By [Jaap, prop. 3.2], it suffices to show that *Mod*($ALat$) has all small products and equalizers. We show first that *Mod*($ALat$) has finite products and equalizers. The terminal object in *Mod*($ALat$) is $(1_{Set} = \{\star\}, 1_{ALat} = \{\star'\}, E_1)$, where $E_1(\star) = \{\star'\}$. It is clearly modest and terminal. The binary product in *Mod*($ALat$) of the modest objects $(X, \mathcal{A}, E_X)$ and $(Y, \mathcal{B}, E_Y)$ is the object $(X \times Y, \mathcal{A} \times \mathcal{B}, E_{X\times Y})$, where $\mathcal{A} \times \mathcal{B}$ is the product of $\mathcal{A}$ and $\mathcal{B}$ in $ALat$ and $E_{X\times Y}(x, y) = E_X(x)\times E_Y(y)$. This means that *Mod*($ALat$) has all finite products. The generalization to all small limits is easy: The product of $(X_i, \mathcal{A}_i, E_i)_{i\in I}$ is $(\Pi X_i, \Pi\mathcal{A}_i, E_I)$, where ΠA_i is the product space and $E_I((x_i)_{i\in I}) = \Pi E_i(x_i)$. The projections are just the set–theoretic projections, tracked by the projections in $ALat$.

To show that *Mod*($ALat$) has equalizers, assume that $f_1, f_2 : (X, \mathcal{A}, E_X) \to (Y, \mathcal{B}, E_Y)$ are two parallel functions, tracked by g_1 and g_2, respectively. To construct an equalizer $(Z, \mathcal{C}, E_Z) \rightarrowtail (X, \mathcal{A}, E_X)$, we must have that Z is the equalizer in *Set* of f_1, f_2. Therefore, we set $Z = \{x \in X | f_1(x) = f_2(x)\}$, $\mathcal{C} = \mathcal{A}$ and E_Z to E_X restricted to this subset. Then the equalizer is (like in *Set*) the obvious inclusion map (which is tracked by the identity). This proves the theorem. □

Note that when we construct the products, equalizers, etc. in *Mod*($ALat$), we always make use of our knowledge of the corresponding constructions in *Set*. Afterwards, we must make sure that all the functions under consideration are in fact realized. This remark also applies when we construct pullbacks (which we now know exist).

Assume that $f : (X, \mathcal{A}, E_X) \to (Z, \mathcal{C}, E_Z)$ (tracked by g_f) and $h : (Y, \mathcal{B}, E_Y) \to (Z, \mathcal{C}, E_Z)$ (tracked by g_h) are two morphisms in *Mod*($ALat$) with the same codomain, and let us construct the pullback of the two. We then draw the following diagram:

$$\begin{tikzcd}
(P, \mathcal{A}_P, E_P) \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"'] & (Y, \mathcal{B}, E_Y) \arrow[d, "h"] \\
(X, \mathcal{A}, E_X) \arrow[r, "f"'] & (Z, \mathcal{C}, E_Z)
\end{tikzcd}$$

We must have that the set P is the pullback of f and h in *Set*. Therefore, we have $P = \{(x, y) \in X \times Y | f(x) = h(y)\}$, and that π_X and π_Y are just the projections. If we set $\mathcal{A}_P = \mathcal{A}\times\mathcal{B}$, and $E_P(x, y) = E_X(x) \times E_Y(y)$, it is easy to see that these are tracked by the projections in $ALat$. Now, if $l : (W, \mathcal{D}, E_W) \to (X, \mathcal{A}, E_X)$ (tracked by g_l) and $m : (W, \mathcal{D}, E_W) \to (Y, \mathcal{B}, E_Y)$ (tracked by g_m) are two arrows with $fl = hm$, then the unique arrow $(W, \mathcal{D}, E_W) \to (P, \mathcal{A}_P, E_P)$ is $\langle l, m\rangle$, tracked by $\langle g_l, g_m\rangle$ (product morphisms).

Naturally, *Mod*($ALat$) is also cartesian closed.

Theorem 9.5. Mod(ALat) *is a cartesian closed category.*

PROOF: We already know this from theorem 8.4 and 7.4, but it is a good exercise to show it directly. We aim at showing that $\mathit{Mod}(\mathit{ALat})$ satifies the requirement on [Jaap, p. 64]. We know that it has finite products, so we are to define exponentials in a suitable way.

For two objects $(X, \mathcal{A}, E_X)$ and $(Y, \mathcal{B}, E_Y)$ in $\mathit{Mod}(\mathit{ALat})$, we define the exponential $(Z, \mathcal{B}^{\mathcal{A}}, E)$, to have

- $Z = \{f : X \to Y | \exists g : A \to B : g \text{ tracks } f\}$
- $\mathcal{B}^{\mathcal{A}}$ the exponential $\{g : A \to B | g \text{ is continuous}\}$ of $\mathcal{A}$ and $\mathcal{B}$ in ALat
- $E(f)$ equal to the elements of $\mathcal{B}^{\mathcal{A}}$ that track f.

To show that this is the exponential, we have to show that there is an arrow $\varepsilon : (Z, \mathcal{B}^{\mathcal{A}}, E) \times (X, \mathcal{A}, E_X) \to (Y, \mathcal{B}, E_Y)$ such that for every object $(W, \mathcal{D}, E_W)$ and map $f : (W, \mathcal{D}, E_W) \times (X, \mathcal{A}, E_X) \to Y$ ()tracked by g), there is a unique morphism $\overline{f} : (W, \mathcal{D}, E_W) \times (X, \mathcal{A}, E_X) \to (Z, \mathcal{B}^{\mathcal{A}}, E)$ such that the following diagram commutes:

$$\begin{tikzcd}
(W \times X, \mathcal{A} \times \mathcal{B}, E_{W \times X}) \arrow[rr, "f"] \arrow[dr, swap, "\overline{f}\times \mathrm{id}"] & & (Y, \mathcal{B}, E_Y) \\
& (Z, \mathcal{B}^{\mathcal{A}}, E) \arrow[ur, swap, "\varepsilon"] &
\end{tikzcd}$$

We define this in the most natural way: set $\varepsilon(f, x) = f(x)$, and for $f : W \times X \to Y$, set $\overline{f} : W \to Y^X$ to $\overline{f}(w) = f(w, -) \in Y^X$. Then ε is tracked by the evaluation map in ALat, and $\overline{f}$ is tracked by $\overline{g}$, also from ALat. □

We wish to show that $\mathit{Mod}(\mathit{ALat})$ is a regular and regular well–powered category. For this, we need to know what the monomorphisms, the epimorphisms, the regular monos, and the regular epis are. This will appear in the next two results.

Proposition 9.6. *A morphism* $f : (X, \mathcal{A}, E_X) \to (Y, \mathcal{B}, E_Y)$ *(tracked by* g_f*) in* Mod(ALat) *is a monomorphism if and only if* $f : X \to Y$ *is injective; it is an epimorphism if and only if* f *is surjective.*

PROOF: Assume first that f is injective, and suppose $f_1, f_2 : (Z, \mathcal{C}, E_Z) \to (X, \mathcal{A}, E_X)$ (tracked by g_1, g_2) are morphisms such that $ff_1 = ff_2$ in $\mathit{Mod}(\mathit{ALat})$. Then $ff_1 = ff_2$ in Set, and therefore $f_1 = f_2$ in Set and also in $\mathit{Mod}(\mathit{ALat})$. For the other direction, assume that f is a monomorphism in $\mathit{Mod}(\mathit{ALat})$, and assume that $f(x_1) = f(x_2)$ for two distinct elements $x_1, x_2 \in X$. Let $a_1 \in E_X(x_1), a_2 \in E_X(x_2)$. We construct the object $(\{\star\}, \mathcal{A}, E_{\{\star\}}) \in \mathit{Mod}(\mathit{ALat})$ where $E_{\{\star\}}(\star) = A$. Then there are two morphisms $f_1 : \star \mapsto x_1$ and $f_2 : \star \mapsto x_2$, tracked by the constant mappings $- \mapsto a_1$ and $- \mapsto a_2$. These morphisms have $ff_1 = ff_2$ in $\mathit{Mod}(\mathit{ALat})$, so $f_1 = f_2$, implying that $x_1 = x_2$.

If f is surjective, then it is easy to see that f is an epimorphism, so assume that f is an epimorphism, and that for some $y_0 \in Y$, there is no $x \in X$ with $f(x) = y_0$. We then construct the object $(\{0,1\}, \mathcal{B}, E_2) \in \mathit{Mod}(\mathit{ALat})$ where $E_2(0) = E_Y(y_0)$ and $E_2(1) = B \setminus E_Y(y_0)$ and two morphisms $f_1, f_2 : (Y, \mathcal{B}, E_Y) \to (\{0,1\}, \mathcal{B}, E_2)$. f_1 is just the constant map with value 1 (tracked by any constant map with a value outside $E_Y(y_0)$). f_2 is 1 everywhere but at y_0 where it is 0. This function is tracked by the identity. Note that $f_1 f = f_2 f$, so by assumption $f_1 = f_2$. This is a contradiction. $\square$

We now move on to characterizing regular epis and monos. This theorem is a bit technical, but it will be useful when we show that *Mod*(*ALat*) is regular and regular well–powered.

Theorem 9.7. *In* Mod(ALat)*, a monomorphism* $m : (X, \mathcal{A}, E_X) \rightarrowtail (Y, \mathcal{B}, E_Y)$ *is regular if and only if it can be written as*

$$\begin{tikzcd}(X, \mathcal{A}, E_X) \arrow[r, "\varphi"] & (X', \mathcal{B}, E_{X'}) \arrow[r, "m'"] & (Y, \mathcal{B}, E_Y)\end{tikzcd}$$

where φ *is an isomorphism,* m' *is realized by the identity, and where* $E_{X'}$ *is the restriction of* E_Y *to the image* $m(X')$ *(so that* $E_X(x) = E_Y(m(x))$*). An epimorphism* $e : (X, \mathcal{A}, E_X) \twoheadrightarrow (Y, \mathcal{B}, E_Y)$ *is regular if and only if it can be written as*

$$\begin{tikzcd}(X, \mathcal{A}, E_X) \arrow[r, "e'"] & (X', \mathcal{A}, E_{X'}) \arrow[r, "\varphi"] & (Y, \mathcal{B}, E_Y)\end{tikzcd}$$

where e' *is realized by the identity,* φ *is an isomorphism and where*

$$E_{X'}(x') = \bigcup_{e(x)=x'} E_X(x).$$

Proof: Assume first that $m : (X, \mathcal{A}, E_X) \rightarrowtail (Y, \mathcal{B}, E_Y)$ is a regular monomorphism, tracked by g_m. Then m fits into an equalizer diagram

$$\begin{tikzcd}(X, \mathcal{A}, E_X) \arrow[r, "m"] & (Y, \mathcal{B}, E_Y) \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & (Z, \mathcal{C}, E_Z)\end{tikzcd}$$

If we can construct another equalizer $m' : (X', \mathcal{A}', E_{X'})$ for f_1, f_2, then it will follow that $(X, \mathcal{A}, E_X)$ and $(X, \mathcal{A}', E_{X'})$ are isomorphic. We do this based on the knowledge we have from *Set*, so we first construct the object $(X' = \{y \in Y | f_1(y) = f_2(y)\}, \mathcal{A}' = \mathcal{B}, E_{X'})$, where $E_{X'}(y) = E_Y(y)$ (so $E_{X'}$ is the restriction of E_Y). Then the inclusion map (call it m') is tracked by the identity and a coequalizer for f_1, f_2:

$$\begin{tikzcd}(X', \mathcal{B}, E_{X'}) \arrow[r, tail, "m'"] & (Y, \mathcal{B}, E_Y) \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & (Z, \mathcal{C}, E_Z) \\ (W, \mathcal{D}, E_W) \arrow[u, dotted, "\langle h, g_h \rangle"] \arrow[ur, "\langle h, g_h \rangle"'] & & \end{tikzcd}$$

This means that m can be written on the desired form.

For the other direction, we just take the case where φ is the identity. So we have a morphism

$$\begin{tikzcd}(X', \mathcal{B}, E_{X'}) \arrow[r, "m'"] & (Y, \mathcal{B}, E_Y)\end{tikzcd} ,$$

where m' is tracked by the identity and $E_{X'}$ is the restriction of E_Y to the image of X' under m'. We aim to show that m' is the equalizer of something. For this, we construct an object $(\{0,1\}, \mathcal{B}, E_2)$ (where $E_2(1) = E_Y(m'(X)) = \bigcup_{y \in m'(X)} E_Y(y), E_2(0) = B \setminus E_2(1))$[3] in *Mod*(*ALat*) and two morphisms

$$\begin{tikzcd}(Y, \mathcal{B}, E_Y) \arrow[r, shift left, "\langle \mathbf{1}_{m'(X)}, \mathrm{id} \rangle"] \arrow[r, shift right, "\langle \mathbf{1}, - \mapsto b_0 \rangle"'] & (\{0,1\}, \mathcal{B}, E_2)\end{tikzcd}$$

Here, b_0 is any element in $E_Y(m'(X))$. We claim that m' is the equalizer for these two morphisms. Clearly, $\mathbf{1}m' = \mathbf{1}_{m'(X)}m'$, and if $\langle h, g_h \rangle : (Z, \mathcal{C}, E_Z) \to (Y, \mathcal{B}, E_Y)$ also has this property, then $(m')^{-1}h$ is the unique function with $h = m'(m')^{-1}h$ (we use that $(m')^{-1}$ is well–defined on the image of h by assumption):

$$\begin{tikzcd}(X', \mathcal{B}, E_{X'}) \arrow[r, "m'"] & (Y, \mathcal{B}, E_Y) \\ (Z, \mathcal{C}, E_Z) \arrow[u, dotted, "(m')^{-1}h"] \arrow[ur, "h"'] & \end{tikzcd}$$

What we need to show is that $(m')^{-1}h$ is in fact a morphism, i.e. that it is realized. For this we note that if $y \in m'(X)$ then $b \in E_Y(y)$ implies $b \in E_{X'}((m')^{-1}(y))$, and from this it follows that g_h tracks $(m')^{-1}h$.

Next, assume that $e : (X, \mathcal{A}, E_X) \to (Y, \mathcal{B}, E_Y)$ is a regular epimorphism, tracked by g_e; it thus fits in a coequalizer diagram

$$\begin{tikzcd}(Z, \mathcal{C}, E_Z) \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & (X, \mathcal{A}, E_X) \arrow[r, "e"] & (Y, \mathcal{B}, E_Y)\end{tikzcd} ,$$

where f_i is tracked by $g_i, i \in \{1,2\}$. The coequalizer of f_1, f_2 in *Set* is made by first considering the least equivalence relation $\sim$ on Y containing $\sim'$, which is defined by

$$y \sim' y' \iff \exists x \in X : f_1(x) = y, f_2(x) = y'$$

Then the coequalizer is the quotient map $Y \to Y/\sim$. If we set $E_\sim([y]) = \bigcup_{y' \in [y]} E_Y(y')$, then the same quotient map is tracked by the identity and is the coequalizer in *Mod*(*ALat*) of f_1, f_2. Any other coequalizer is isomorphic to this one, as before, and can thus be written on the desired form.

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[3] Note that this is only well–defined if m' is not surjective. However, it is easy to show that if this is the case, m' is an isomorphism and therefore trivially an equalizer.

For the reverse implication, we just take the case where $\varphi = \mathrm{id}$. So we have an epimorphism

$$\begin{tikzcd}(X,\mathcal{A},E_X) \arrow[r, twoheadrightarrow, "\langle e,\mathrm{id}\rangle"] & (Y,\mathcal{A},E_Y)\end{tikzcd}$$

where $E_Y(y) = \bigcup_{e(x)=y} E_X(x)$. We are to show that e is a coequalizer. For this we consider the equivalence relation $\sim$ on X defined by

$$x \sim x' \iff e(x) = e(x')$$

Let $X/\sim$ be the quotient, and use the Axiom of Choice to pick out a set $X_\sim \subseteq X$ of representatives for each equivalence class. We then construct the set $Z = \{(x,\underline{x}) \in X \times X_\sim | x \in [\underline{x}]\}$ and the object in *Mod*(*ALat*) given by $(Z, \mathcal{A}\times\mathcal{A}, E)$, where $E(x,\underline{x}) = E_X(x) \times E_X(\underline{x})$. Then the two projections

$$\begin{tikzcd}(Z,\mathcal{A}\times\mathcal{A},E) \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & (X,\mathcal{A},E_X)\end{tikzcd}$$

are tracked by the projections in *ALat*, and we claim that e is the coequalizer of the two morphisms. Certainly, $e\pi_1 = e\pi_2$, so suppose that we have another morphism h with this property:

$$\begin{tikzcd}(Z,\mathcal{A}\times\mathcal{A},E) \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & (X,\mathcal{A},E_X) \arrow[r, twoheadrightarrow, "\langle e,\mathrm{id}\rangle"] \arrow[dr, "\langle h,g_h\rangle"'] & (Y,\mathcal{A},E_Y) \arrow[d, dotted, "j"] \\ & & (W,\mathcal{D},E_W)\end{tikzcd}$$

We are to uniquely define the function $j : Y \to W$ such that $je = h$, and show that it has a realizer. Define j by

$$j(y) = h(x) \quad \text{for any } x \in X \text{ with } e(x) = y$$

Since $h\pi_1 = h\pi_2$, this definition makes sense and it is our only choice. We claim that g_h realizes j. To show this, let $y \in Y$, and let $x \in X$ satisfy $e(x) = y$. Also, let $a \in E_Y(y) = \bigcup_{x' \in [x]} E_X(x')$. This means that there exists $x_0 \in [x]$ so that $a \in E_X(x_0)$, implying that $g_h(a) \in E_W(h(x_0))$. But since $e(x_0) = y$, we get that $h(x_0) = j(y)$, so we have that $g_h(a) \in E_W(j(y))$, as desired. This completes the proof of the theorem. $\square$

We are now ready to show that *Mod*(*ALat*) is regular. Recall the definition of a regular category, for example in [Jaap, ch. 4].

Theorem 9.8. Mod(ALat) *(and thereby* EQU*) is a regular category.*

Proof: All that is left to show is that regular epimorphisms in *Mod*(*ALat*) are stable under pullback. So assume that we have a pullback square

$$\begin{tikzcd}(P,\mathcal{A}_P,E) \arrow[r] \arrow[d, "e_P"'] & (X,\mathcal{B},E_X) \arrow[d, twoheadrightarrow, "e"] \\ (Z,\mathcal{C},E_Z) \arrow[r, "f"'] & (Y,\mathcal{B},E_Y)\end{tikzcd}$$

where e is a regular epimorphism. We have just taken the simple case, where e is realized by the identity and $E_Y(y) = \bigcup_{e(x)=y} E_X(x)$ (so that e is a "canonical" representative for the quotient[4] that it belongs to). We are to show that e_P is a regular epimorphism. We have already considered pullbacks, so we can reason about the pullback object. In fact, $P = \{(z,x) \in Z \times X | f(z) = e(x)\}$, $\mathcal{A}_P = \mathcal{C} \times \mathcal{B}$, and $E(z,x) = E_Z(z) \times E_X(x)$. Also, we know that e_P is just the projection $e_P : P \to Z$, tracked by the projection $\pi_{\mathcal{C}} : \mathcal{C} \times \mathcal{B} \to \mathcal{C}$. Since e is surjective, e_P is also surjective. We can write e_P as a composite:

$$\begin{tikzcd}(P, \mathcal{C} \times \mathcal{B}, E_P) \arrow[r, two heads, "e'_P"] & (Z, \mathcal{C} \times \mathcal{B}, E'_Z) \arrow[r, "\mathrm{id}"] & (Z, \mathcal{C}, E_Z)\end{tikzcd}$$

Here, e'_P is the same set–theoretical projection, and $E'_Z(z) = E_Z(z) \times \mathcal{B}$. Then e'_P is tracked by the identity on $\mathcal{C} \times \mathcal{B}$, and $\mathrm{id} : Z \to Z$ is tracked by the second projection (and one can easily show that it is an isomorphism). By the previous theorem, this means that e_P is a regular epimorphism, as required. ☐

With the characterization of regular monos that we get from theorem 9.7, we can also show that *Mod*(*ALat*) (and thus *EQU*) is regular well–powered. This is an easy corollary to the following.

Theorem 9.9. *The regular subobjects of an object* $(Y, \mathcal{B}, E_Y)$ *in* Mod(ALat) *are in bijective correspondence with* $\mathcal{P}(X)$.

Proof: By the characterization of regular monos, every regular subobject $(X, \mathcal{A}, E_X) \rightarrowtail (Y, \mathcal{B}, E_Y)$ has a "canonical" representative $m : (X', \mathcal{B}, E_{X'}) \to (Y, \mathcal{B}, E_Y)$ where m is tracked by the identity, and $E_{X'}$ is the restriction of E_Y to the image of X. We aim to construct a "supercanonical" representative, so we construct the object $(m(X), \mathcal{B}, E)$ where $E(m(x)) = E_Y(m(x))$. Then the inclusion i of $m(X)$ into Y is tracked by the identity, and it is easy to see that $(X, \mathcal{B}, E_X)$ and $(m(X), \mathcal{B}, E)$ are isomorphic. This means that the inclusion of the image of a regular mono represents the same regular subobject as the regular mono. Any subset of $Y' \subseteq Y$ represents exactly one regular subobject, namely the inclusion $i : (Y', \mathcal{B}, E_{Y'}) \to (Y, \mathcal{B}, E_Y)$, where $E_{Y'}$ is just the restriction of E_Y. It is obvious that these two operations are mutually inverse; this proves the statement. ☐

10 Conclusion and Future Work

This project is the last thing I need to qualify for the Ph.D. scholarship that I have already received. I am certain that my work with this project will help me a lot later on, and that I have just begun to scratch the surface of the world of Domain Theory. In particular, I wish to complete my understanding of [Ba],

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[4] A *quotient* is the dual notion of a subobject.

so that I can begin to understand concepts related to models of dependent type theory and realizability.

Looking at the project, I feel that I have learned a lot and done a lot of work in spelling out details where they are left to the reader in [DS98] and [Ba], and I think that it is conveyed in the resulting paper. Alas, I didn't get all the way to the model of dependent type theory that I hinted at in the project description, but I feel that it was better to take a more realistic approach (and showing regularity) and then understand this subject in full detail.

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A Project Description

The aim of this project is to understand and work with concepts dealing with the subject of Domain Theory so that we can grasp the contents of the article [DS98]. In order to do this, we first study the Course notes [DSn] from a Ph.D. course held at Carnagie Mellon University, Pensylvania by Prof. Dana Scott. For this however, we may need to consult [DP] for neccesary definitions. Both [DSn] and [DP] contain numerous exercises, and we document our understanding of the material by solving the most important of these. Doing so enables us to work with concepts such as lattices, order, domains, and models so that we have the tools to read and understand [DS98] and [Ba].

Informally, the main idea in [DS98] and [Ba] is as follows: The category TOP of topological spaces and continuous mappings does not have the property of cartesian closure, which is a problem, since this property is needed for modelling e.g. the λ-calculus. Traditionally, there have been two trends in the mending of this problem:

- Consider only *compactly generated spaces.*
- Expanding TOP to certain kinds of limit spaces.

The two articles introduce yet another solution that is in a sense more natural, namely that of *equilogical spaces*, which in a natural way contains TOP. This category is introduced and it is shown that it is indeed cartesian closed. Moreover, it is shown in [Ba] how this category can be used to model dependent type theory. This last subject is highly advanced and at this point we believe that the project will not get into all details here.