Bits of Pitts

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Alternate proof of Theorem 4.16: By Lemma 4.14 we have that $\mathscr{R}_{adm}(D)$ is a complete lattice with meets given by the intersections of Definition 4.12, let $\perp$ and $\top$ denote the least respectively greatest element of this lattice. We define ψ and $\psi^{\S}$ monotone as in the original proof, and copy the search for a relation $\Delta \in \mathscr{R}_{adm}(D)$ such that $\psi(\Delta, \Delta) = \Delta$ holds. We construct

$$\Delta^- = \bigsqcup_{n \in \mathbb{N}} \pi_1 \left((\psi^{\S})^n (\perp, \top) \right)$$

and

$$\Delta^+ = \bigcap_{n \in \mathbb{N}} \pi_2 \left((\psi^{\S})^n (\perp, \top) \right)$$

as joins respectively meets in $\mathscr{R}_{adm}(D)$. We now aim to prove that $\Delta^- = \Delta^+$ and that this relation is the desired one.

Notice initially that in the complete lattice $\mathscr{R}_{adm}(D)^{op} \times \mathscr{R}_{adm}(D)$ the element (Δ^-, Δ^+) constitutes the meet of the set $\left\{ (\psi^{\S})^n (\perp, \top) \mid n \in \mathbb{N} \right\}$ and that $(\perp, \top)$ is least element. Combining this with the monotonicity of $\phi^{\S}$ we get by purely order theoretic considerations that $\psi^{\S}(\Delta^-, \Delta^+)$ is less than or equal to (Δ^-, Δ^+) in $\mathscr{R}_{adm}(D)^{op} \times \mathscr{R}_{adm}(D)$, i.e., that $\psi(\Delta^+, \Delta^-) \supset \Delta^-$ and that $\psi(\Delta^-, \Delta^+) \subset \Delta^+$.

It now remains to prove $\Delta^- = \Delta^+$ and the $\Delta^- \subset \Delta^+$ part of this equality is obtained also by order theoretic arguments: Proving $\Delta^- \subset \Delta^+$ comes down to proving

$$\pi_1 \left((\psi^{\S})^{n_1} (\perp, \top) \right) \subset \pi_2 \left((\psi^{\S})^{n_2} (\perp, \top) \right)$$

for arbitrary $n_1, n_2 \in \mathbb{N}$. But by monotonicity of $\psi^{\S}$ we get that

$$\pi_1 \left((\psi^{\S})^{n} (\perp, \top) \right) \subset \pi_1 \left((\psi^{\S})^{n+1} (\perp, \top) \right)$$

and that

$$\pi_2 \left((\psi^{\S})^{n} (\perp, \top) \right) \supset \pi_2 \left((\psi^{\S})^{n+1} (\perp, \top) \right)$$

for arbitrary $n \in \mathbb{N}$. And as we by monotonicity of ψ also have

$$\pi_1 \left((\psi^{\S})^{n} (\perp, \top) \right) \subset \pi_2 \left((\psi^{\S})^{n} (\perp, \top) \right)$$

for arbitrary $n \in \mathbb{N}$ we are done.

Finally we need to prove $\Delta^+ \subset \Delta^-$. The crucial insight here – indeed of the entire proof – is that we have

$$\delta^n(\perp) : \pi_2 \left((\psi^{\S})^{n} (\perp, \top) \right) \subset \pi_1 \left((\psi^{\S})^{n} (\perp, \top) \right)$$

for all $n \in \mathbb{N}$ where δ is from Definition 3.2 and the leftmost $\perp$ denotes the constant bottom function on D. Assuming this equality we immediately get $\delta^n(\perp) : \Delta^+ \subset \Delta^+$ for all $n \in \mathbb{N}$ which by admissibility of Δ^+ and the minimal invariant property implies the desired. The equality is proved by induction using the fact that $e : R \subset S$ implies $\delta(e) : \psi(S, R) \subset \psi(R, S)$ for arbitrary $e \in D \multimap D$ and $S, R \in \mathscr{R}_{adm}(D)$, this result is a central consequence of the definition of an admissible action, cf. Definition 4.6. □

One might ask what good another proof does as the original works just fine and as they produce identical relations by Corollary 4.10. It is the opinion of the author that while the above proof is not exactly constructive is has a constructive feel to it that is absent in the original. The operational intuition of taking the $\top$ relation and repeatedly applying ψ to refine it is hidden within the application of the Tarski-Knaster fixed point theorem in the original proof.

A word of warning: We intuitively think of intersections of the abstract relations of Definition 4.1 as ordinary set theoretic intersections and in all cases of Example 4.2 this is just what they are. But while this implies that meets in $\mathscr{R}_{adm}(D)$ are just set theoretic intersections, one cannot deduce that joins are given by set theoretic union, rather it is given by meets of upper bounds. Intuitively it is more fair to think of joins in $\mathscr{R}_{adm}(D)$ as set theoretic unions with just that little bit more added to be admissible.

References

[1] Pitts, A. M.: *Relational Properties of Domains*, Information and Computation **127** (1996), 66-90.