FM-sets and FM-cpos: A 2-Page Excerpt of [2] and [1]

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What follows are the basic definitions and some results on FM-sets, also known as nominal sets. We provide very little in the way of proofs but most – though not all – results are straightforward to verify.

The Basics

Let $\mathbb{A}$ denote a fixed countably infinite set – the set of *atoms* – and $perm(\mathbb{A})$ the bijections on $\mathbb{A}$ that fix all but finitely many atoms. In having only a single sort of atoms we follow [2] but disallowing arbitrary bijections is the approach of [1].

Definition 1 *An* action *on a set D is an operation $\cdot : perm(\mathbb{A}) \times D \to D$ such that we have*

$$\forall d \in D : id \cdot d = d$$

and also have

$$\forall \pi, \pi' \in perm(\mathbb{A}) \forall d \in D : \pi \cdot (\pi' \cdot d) = (\pi \circ \pi') \cdot d.$$

Definition 2 *An* FM-set *is a set D equipped with an action such that any element has a* finite support*, i.e., for any $d \in D$ there is a finite subset $A \subset \mathbb{A}$ with*

$$\forall \pi \in perm(\mathbb{A}) : (\forall a \in A : \pi(a) = a) \Rightarrow \pi \cdot d = d.$$

Any subset of a nominal set that is closed under the action of all permutations naturally becomes a nominal set itself by restriction of the action.

It is a basic fact of group theory that any bijection of a finite set is the finite composition of *transpositions*, i.e., bijections that exchange two elements and fix all other elements. This paves the way to the following:

Proposition 3 *An intersection of two finite supports of an element of an FM-set is again a finite support of that element. Consequently, any element d of an FM-set has a least finite support denoted* $\mathrm{supp}(d)$.

If furthermore this least finite support is empty we say that the element is *equivariant.*

Lemma 4 *Let D be a set equipped with an action $\cdot$. If A is a finite support for some $d \in D$ then $\pi(A)$ is a finite support for $\pi \cdot d$ for any $\pi \in perm(\mathbb{A})$.*

This lemma is useful for proving the following corollary but also because it allows us to build FM-sets from arbitrary sets equipped with an action by restricting to elements with finite support.

Corollary 5 *For any FM-set D we have that*

$$\forall \pi \in perm(\mathbb{A}) \forall d \in D : \mathrm{supp}(\pi \cdot d) = \pi(\mathrm{supp}(d)).$$

Definition 6 *Let D_0 and D_1 be FM-sets. By setting*

$$(\pi \cdot f)(d) = \pi \cdot (f(\pi^{-1} \cdot d))$$

for arbitrary $\pi \in perm(\mathbb{A})$, $f : D_0 \to D_1$ and $d \in D_0$ we define an action on $D_1^{D_0}$. The subset $D_0 \to_{\mathrm{fs}} D_1$ of finitely supported functions *is an FM-set.*

Note that for $f : D_0 \to D_1$, $\pi \in perm(\mathbb{A})$ and $d \in D_0$ we always have the equality

$$\pi \cdot (f(d)) = (\pi \cdot f)(\pi \cdot d).$$

Also it is sometimes useful to know that for $f : D_0 \to D_1$ and $\pi \in perm(\mathbb{A})$ we have that

$$\pi \cdot f = f \iff \forall d \in D_0 : f(\pi \cdot d) = \pi \cdot (f(d)).$$

Not all functions between FM-sets are finitely supported, it is a fun fact that no function $c : (\mathbb{A} \to_{\mathrm{fs}} \mathbb{B}) \to \mathbb{A}$ such that

$$\forall f \in \mathbb{A} \to_{\mathrm{fs}} \mathbb{B} \forall a \in \mathbb{A} : f(a) = true \Rightarrow f(c(f)) = true$$

can be finitely supported. Here $\mathbb{A}$ comes equipped with the action of function application and $\mathbb{B} = \{true, false\}$ is equipped with the *discrete action*, i.e., the projection of the second argument.

Proposition 7 *The category of FM-sets and equivariant functions is cartesian closed with the finitely supported functions as exponent and the remaining constructions as in the category of sets and functions.*

Definition 8 *Let D be FM-set. By setting*

$$\pi \cdot D' = \{\pi \cdot d \mid d \in D'\}$$

for arbitrary for $\pi \in perm(\mathbb{A})$ and $D' \subset D$ we define an action on $\mathcal{P}(S)$, the powerset of D. The subset $\mathcal{P}_{\mathrm{fs}}(D)$ of finitely supported subsets *is an FM-set.*

Note that for $D' \subset D$ and $\pi \in perm(\mathbb{A})$ we have $\pi \cdot D' = D'$ if and only if $d \in D' \Leftrightarrow \pi \cdot d \in D'$ for all $d \in D$, this is useful when working with relations.

Nominal Domain Theory

Definition 9 *An* FM-cpo *is an FM-set D equipped with an equivariant partial order $\sqsubseteq$ such that any ascending chain has a least upper bound, provided that the union of the least finite supports of its elements is itself finite.*

We refer to the last property by saying that the *chain is finitely supported*. Note that equivariancy by Definition 8 means that

$$d_0 \sqsubseteq d_1 \iff \pi \cdot d_0 \sqsubseteq \pi \cdot d_1.$$

holds for all $\pi \in perm(\mathbb{A})$ and all $d_0, d_1 \in D$.

Proposition 10 *Let D be an FM-cpo and let $(d_n)_{n\in\mathbb{N}}$ be a finitely supported chain in D. We have*

$$\forall \pi \in perm(\mathbb{A}) : \pi \cdot \bigsqcup_{n\in\mathbb{N}} d_n = \bigsqcup_{n\in\mathbb{N}} \pi \cdot d_n.$$

Definition 11 *Let D_0 and D_1 be FM-cpos. A function $f \in D_0 \to_{\mathrm{fs}} D_1$ is called* monotone *if*

$$\forall d_0, d_1 \in D_0 : d_0 \sqsubseteq d_1 \Rightarrow f(d_0) \sqsubseteq f(d_1)$$

and continuous *if it is monotone and we have*

$$f\left(\bigsqcup_{n\in\mathbb{N}} d_n\right) = \bigsqcup_{n\in\mathbb{N}} f(d_n)$$

for all finitely supported chains $(d_n)_{n\in\mathbb{N}}$ in D_0.

Proposition 12 *Let D be a pointed FM-cpo. Any continuous function $f \in (D \to D)$ has a least fixed point* $\mathrm{fix}(f) = \sqcup_{n\in\mathbb{N}} f^n(\bot)$.

Proposition 13 *The FM-set of finitely supported continuous functions ordered pointwise is an FM-cpo. The category of FM-cpos and equivariant continuous functions is cartesian closed with this exponent.*

References

[1] Pitts, A. M.: *Alpha-Structural Recursion and Induction*, Journal of the ACM **53** (2006), 459-506.

[2] Shinwell, M. R.: *The Fresh Approach: Functional Programming with Names and Binders*, Ph.D. Thesis, University of Cambridge, 2005.