Max Power: A Pretty Simple Flow Invariant

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The Object of our Desires

What would we like?

An invariant to tell subshifts generated by aperiodic primitive substitutions apart if they are not flow equivalent.

- We would like the invariant to be easily computable...
- ...and values of the invariant to be easily comparable.
- And of course we would like the invariant to be *strong*, i.e., it should tend to give different values when flow equivalence fails.

While completeness is desirable as well, it is probably a little too much to ask for.

How do we get it, if we get it at all?

It turns out that just by taking the least upper bound of the number of repetitions of nonempty words – including partial repeats – in the language of the substitution we come a long way towards this goal.

The Building Blocks

Finite – possibly non integer – repetitions as periods

A *period* p of a subshift X is an ordered pair $(u, n) \in \mathcal{L}(X) \times \mathbb{N}$ with $|u| \geq n$ such that $u_{[i]} = u_{[i+n]}$ for all $i \in \{0, \ldots, |u| - n - 1\}$. We refer to n as the *length* of p and to $|u|/n$ as the *power* of p, these are denoted $|p|$ respectively $P(p)$.

We study repetitions of words in the language of the subshift as periods, the power of a period is the — possibly non integer — number of repetitions. A period could be $(\mathtt{ababa}, 2)$ if $\mathtt{ababa}$ occurs in the language, this has power $5/2$.

What was a substitution again?

A *substitution* τ over the alphabet $\mathcal{A}$ is a map $\tau : \mathcal{A} \to \mathcal{A}^+$, it can be extended by concatenation to a map $\tau : \mathcal{A}^* \to \mathcal{A}^*$. We demand *primitivity* of all our substitutions, this eliminates undesired substitutions. A primitive substitution τ has the language $\mathcal{L}(\tau)$ of all subwords of $\tau^n(\alpha)$ for arbitrary $\alpha \in \mathcal{A}$ and $n \in \mathbb{N}$ and it generates the subshift X_τ over $\mathcal{A}$ of all sequences of $\mathcal{A}^{\mathbb{Z}}$ with all subwords in $\mathcal{L}(\tau)$.

Building with the Blocks

The Max Power Theorem

A primitive substitution generating an infinite subshift is *aperiodic*, Mossé showed that the power of the periods of one such subshift has an upper bound. For an aperiodic primitive substitution τ we may thus define Max Power as the supremum of the powers of the periods of X_τ. And this is an invariant of flow equivalence — some of the time: *Any two fair aperiodic primitive substitutions that generate flow equivalent subshifts have identical Max Powers.*

What it takes to be fair

A primitive substitution τ is said to be *fair* if for any period p of X_τ and any $N \in \mathbb{N}$ there exists a period q of X_τ with $P(q) \geq P(p)$ and $|q| \geq N$.

The natural way of obtaining q from p is by application of the substitution, we do, e.g., have fixed length primitive substitutions fair.

Suspense Stunt / mimicking Carlsen and Eilers

The contestants

τ : $\mathtt{a} \mapsto \mathtt{accdadbb}$, $\mathtt{b} \mapsto \mathtt{acdcbadb}$, $\mathtt{c} \mapsto \mathtt{aacdcdbb}$, $\mathtt{d} \mapsto \mathtt{accbdadb}$.

ν : $\mathtt{a} \mapsto \mathtt{accbbadd}$, $\mathtt{b} \mapsto \mathtt{accdbabd}$, $\mathtt{c} \mapsto \mathtt{aacbbcdd}$, $\mathtt{d} \mapsto \mathtt{acbcdabd}$.

The claim

Carlsen and Eilers proved the subshifts generated by these aperiodic primitive substitutions not to be flow equivalent by means of a cleverly constructed, advanced invariant partly originating in the world of operator algebra. Here we shall reproduce this particular result live using Max Power and cheating not overly much.

Proving the Max Power Theorem

Flow equivalence as conjugacy and symbol expansion

Flow equivalence is generated by conjugacy and symbol expansion, we thus ponder the behavior of periods under these constructions. One shows that a sufficiently long period may be moved across either a conjugacy or a symbol expansion with arbitrarily small loss of power and with the new period arbitrarily long.

What's the frequency, Max?

This moving of periods is easy in the case of conjugacy, symbol expansions on the other hand are slightly troublesome as, e.g., the period $(\mathtt{aba}, 2)$ with power $3/2$ could be symbol expanded to $(\mathtt{abca}, 3)$ with power $4/3$. This is solved by long periods as the frequency of each letter converges as we consider longer words. Frequencies exist as a consequence of Perron-Frobenius; we have to show that their existence is preserved by conjugacy and symbol expansion.

Looking for Max: Two general tools, one application

Maximally extended periods with surroundings: Hedged periods

The word aa occurs in abaab $= \tau_F^3(\mathtt{a})$ where τ_F is the aperiodic primitive substitution $\mathtt{a} \mapsto \mathtt{ab}$, $\mathtt{b} \mapsto \mathtt{a}$ called the *Fibonacci substitution*. This makes $(\mathtt{aa}, 1)$ a maximally extended period of X_{τ_F}. But applying τ_F yields $(\mathtt{abab}, 2)$ which is not maximally extended in abaababa $= \tau_F^4(\mathtt{a})$. So we need to consider periods with a bit of surroundings as well, this is captured in the slightly technical definition of *hedged periods*, we skip the details here.

Inverse hedged periods

Thanks to Mossé it is the case that hedged periods of sufficient length and power are images of other hedged periods under application of the substitution. This – in some sense – reduces our quest for power to a finite set of hedged periods.

An exact value

Specialized considerations now give $\mathrm{MP}(\tau_F) = \varphi + 2$, where φ is the golden mean.

Suspense Stunt / mimicking Carlsen and Eilers, part II

And the winners is...

Let us have a look at our calculations.

Summing up: Are we satisfied?

What were our goals again:

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- ...and values of the invariant to be easily comparable.
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While completeness is desirable as well, it is probably a little too much to ask for.

Did we meet our goals?

To some degree – I hope that you agree that we did not fail them completely.
Thank you for your attention.