Flow Equivalence between Substitutional Dynamical Systems

Master's thesis presentation, Jacob Thamsborg

August 24th 2006

Abstract

We concern ourselves with the what and when of flow equivalence between subshifts generated by aperiodic primitive substitutions. Equivalent definitions of flow equivalence are stated and we discuss the two invariants Max Power and weak equivalence between such subshifts.

Shifts, subshifts and symbol expansion

For an alphabet $\mathcal{A}$ we consider the topological space $\mathcal{A}^{\mathbb{Z}}$ equipped with the product topology of the discrete topology on $\mathcal{A}$. We denote by $T : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ the *shift* on $\mathcal{A}^{\mathbb{Z}}$, i.e., $(T(x))_p = x_{p+1}$ for any $x \in \mathcal{A}^{\mathbb{Z}}$ and $p \in \mathbb{Z}$. This is a homeomorphism.

We call a subset $X \subset \mathcal{A}^{\mathbb{Z}}$ a *subshift* over $\mathcal{A}$ if it is closed and invariant under application of T. Any subshift is compact with a basis of clopen sets. To subshifts are said to be *conjugate* if they are homeomorphic in a shift preserving way.

For symbols $\alpha \in \mathcal{A}$ and $\bullet \notin \mathcal{A}$ we denote by $X^{\alpha\bullet}$ the symbol extension of the subshift X, i.e., the sequences of X with all occurrences of α replaced by $\alpha\bullet$ and with the missing links added.

The suspension of a subshift

Definition 1. *For a subshift X we consider the set $X \times \mathbb{R}$ equipped with the product topology and let $\sim$ be the equivalence relation on this set generated by*

$$\{(x, r) \sim (T(x), r - 1) \mid x \in X, r \in \mathbb{R}\}\,.$$

We then define the suspension *of X as the space $X \times \mathbb{R}/\sim$ equipped with the quotient topology and denote it ΣX.*

Note that two pairs (x, r) and (y, s) of $X \times \mathbb{R}$ are related exactly when $r - s \in \mathbb{Z}$ and $y = T^{r-s}(x)$. Intuitively, our desire is to make sense of $T^r(x)$ for $x \in X$ and any $r \in \mathbb{R}$, i.e., to produce a continuous extension of the discrete subshift. Any suspension is Hausdorff and compact.

Flow equivalence of subshifts

Definition 2. *Two subshifts X and Y are said to be flow equivalent if there is a homeomorphism $\Phi : \Sigma X \to \Sigma Y$ such that for any $z \in \Sigma X$ we may choose $\phi_z : \mathbb{R} \to \mathbb{R}$ monotonically increasing with $\Phi(T^r(z)) = T^{\phi_z(r)}(\Phi(z))$ for any $r \in \mathbb{R}$.*

Intuitively we may stretch each orbit at will whilst preserving the orientation. Notice how we make use of our new ability to shift by any real number and not just integers.

It is a classic result that the relation of flow equivalence on subshifts is generated by conjugacy and symbol extension. Proving this in detail was a goal of the thesis, here we shall give the short illustrated proof on two slides.

Om kummefrysere og konjugerethed

Betragt de følgende tre par af varierende versioner af ordet kummefryser:

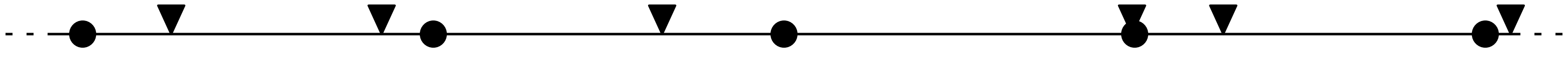
`kummefryser`	`kummefryser`	`kummefryser`
`kummefryser`	`kåmmefryser`	`kumefryser`

De to ord i det første par er de samme. I det midterste par er ordene næsten ens, kun et enkelt bogstav er forkert. I det sidste par er ordene også næsten ens – men på en anden måde end i det midterste par. Her mangler et bogstav, ordene har derfor ikke samme længde og det nederste skal intuitivt *strækkes* for at stemme overens med det øverste.

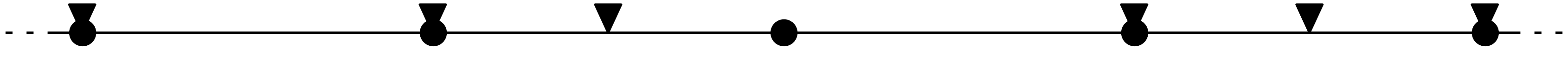
Der er altså *mere end en måde at være ens på*. De tre måder ovenfor svarer løst til det vi kalder lighed, konjugerethed hvv. strømningsækvivalens / flow equivalence.

Proving equivalence of the two definitions - Part I

Consider the following intuitive picture of two flow equivalent subshifts:



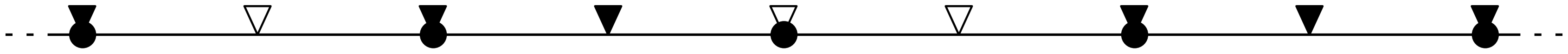
The sequences of the first subshift are placed in order but otherwise rather randomly. We may remedy this to obtain the following situation – essentially by hacking the flow equivalence:



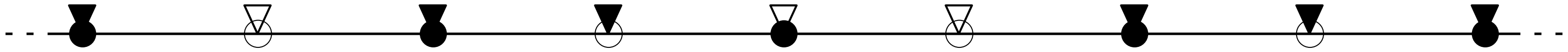
This is not trivial, it is here we produce something discrete from something continuous. It relies on any subshift being compact with a basis of clopen sets; also we use continuity of the *first return time* to preserve order.

Proving equivalence of the two definitions - Part II

We continue our quest by symbol extending appropriate sequences of the first subshift to obtain:



Note how we really cheated above as we need to pass to higher block shifts to be able to identify the appropriate sequences from their zeroth coordinate alone. But this can be done whilst preserving conjugacy and, as such, is okay. Finally we symbol extend every single sequence of the second subshift and we arrive at conjugate subshifts:



Kum(m)efryseren vender tilbage

Vi husker fra tidligere kumefryseren med det manglende m. Lad os prøve at afhjælpe dens problem ved hjælp af symbolekspansion og konjugerethed.

`kumefryser`

er altså startordet og vi foretager symbolekspansion ved at erstatte hvert `m` i ordet med `m`• hvilket giver os det nye ord

`kum`•`efryser`

med den rette længde. Afslutningsvis ændrer vi • til `m` og får som ønsket

`kummefryser`

idet vi bemærker at de to sidste ord er ens på den måde vi tidligere kaldte konjugerethed. Dermed illustrerer eksemplet det fundamentale resultat at symbolekspansion og konjugerethed tilsammen udgør strømningsækvivalens.

Substitutions and the Max Power Theorem - Part I

By a *substitution* τ over the alphabet $\mathcal{A}$ we understand a map $\tau : \mathcal{A} \to \mathcal{A}^+$, this can be extended by concatenation to a map $\tau : \mathcal{A}^* \to \mathcal{A}^*$. We demand *primitivity* of all our substitutions, this is a technical nicety. A primitive substitution has a *language* $\mathcal{L}(\tau)$ consisting of all words that occur in $\tau^n(\alpha)$ for arbitrary $\alpha \in \mathcal{A}$ and $n \in \mathbb{N}$ and it *generates a subshift* X_τ over $\mathcal{A}$ containing sequences of $\mathcal{A}^\mathbb{Z}$ with all occurring words contained in $\mathcal{L}(\tau)$.

By a *period* of a subshift we understand a tuple (u, n) such that $uu^\infty_{[0,n-1]}$ occurs in the language of the subshift, we refer to $|u|$ as the *length* of the period and assign it the *power* $\frac{n}{|u|}$. The idea is to study repetitions in the language of the subshift, a period represents a repetition and its power is the – possibly non integer – number of repetitions. A period of a subshift could be $(ab, 3)$ if $ababa$ occurs in the language of the substitution, this has a power of $\frac{3}{2}$ corresponding to one and a half repetitions.

Substitutions and the Max Power Theorem - Part II

We say that a primitive substitution is aperiodic if it generates an infinite subshift. It is a classical result that any aperiodic primitive substitution generates subshifts with an upper bound on the powers of its periods. For such a substitution τ we may therefore define its *Max Power*, denoted $\mathrm{MP}(\tau)$, as the supremum of the powers of the periods of X_τ. And this is an invariant for flow equivalence – some of the time:

Theorem 1. *Any two fair aperiodic primitive substitutions that generate flow equivalent subshifts have identical Max Powers.*

The *fair* primitive substitutions are the ones that generate subshifts that for any period p and any length $N \in \mathbb{N}$ have a period q with at least as great a power and $|q| \geq N$. The natural way of obtaining q from p is by application of the substitution, we do, e.g., have that all fixed length primitive substitutions are fair.

Proof

Applying the result that flow equivalence is generated by conjugacy and symbol expansion we ponder the behavior of periods under these constructions. More precisely we show that we may move a sufficiently long period across either a conjugacy or a symbol expansion with arbitrarily small loss of power and with the new period arbitrarily long.

This is easy in the case of conjugacy as any shift preserving continuous map is a sliding block code and as such loses a fixed number of symbols. Symbol expansions are slightly troublesome as, e.g., the period $(ab, 1)$ with power $1/2$ could be symbol extended to $(abc, 1)$ with power $1/3$. But this also is solved by long periods as the frequency of each letter converges as we consider longer words. That frequencies exist in subshifts generated by primitive substitutions is a consequence of Perron-Frobenius; we have to show that their existence is preserved by conjugacy and symbol expansion.

Om at at gætte uden at kigge - Del I

På næste slide er – som vi tidligere har oplevet – skrevet tre par af ord. Lad os prøve for hvert par at gætte om de to ord er helt ens, uden at kigge på dem. Som hjælp har vi angivet ordenes længder målt i antal bogstaver herunder:

`længde: 6`	`længde: 5`	`længde: 6`
`længde: 5`	`længde: 5`	`længde: 6`

Lad os overveje hvad vi egentlig kan gætte og hvad vi ikke kan gætte. Og lad os så skifte slide så vi kan se om vi har gættet rigtigt.

Om at at gætte uden at kigge - Del II

De lovede ord er som følger:

`fryser`	`kumme`	`fryser`
`fryse`	`kumme`	`sveder`

Det midterste par bestod altså af ens ord og de andre sidste par af forskellige. Havde vi gættet rigtigt – og især gennemskuet hvor meget vi kunne gætte?

Vi har faktisk introduceret længden af et ord som en *invariant* for lighed og er straks løbet panden mod muren: Hvis invarianterne er forskellige er ordene det også, men hvis invarianterne er ens er vi lige vidt!

Husk samtidig fra tidligere at der er andre måde at være ens på end almindelig lighed, og vi kan tale om invarianter for dem alle, eksempelvis en invariant for strømningsækvivalens.

Weak Equivalence

Definition 3. *Let τ and υ be primitive substitutions over the alphabets $\mathcal{A}$ respectively $\mathcal{B}$. We say that τ and υ are* weakly equivalent, *denoted $\tau \sim_w \upsilon$, if there are sequences of natural numbers $(m_i)_{i\in\mathbb{N}}$ and $(n_i)_{i\in\mathbb{N}}$, a sequence of maps $(\rho_i)_{i\in\mathbb{N}}$ from $\mathcal{A}$ to $\mathcal{B}^+$ and a sequence of maps $(\sigma_i)_{i\in\mathbb{N}}$ from $\mathcal{B}$ to $\mathcal{A}^+$ such that the following infinite diagram commutes:*

$$\begin{tikzcd}
& {} \arrow[dl, "\rho_1"'] & & {} \arrow[ll, "\tau^{m_1}"'] \arrow[dl, "\rho_2"'] & & {} \arrow[ll, "\tau^{m_2}"'] \arrow[dl, "\rho_3"'] & & \cdots \\
{} & & {} \arrow[ll, "\upsilon^{n_1}"] \arrow[ul, "\sigma_1"] & & {} \arrow[ll, "\upsilon^{n_2}"] \arrow[ul, "\sigma_2"] & & {} \arrow[ll, "\upsilon^{n_3}"] \arrow[ul, "\sigma_3"] &
\end{tikzcd}$$

Weak equivalence as an invariant

It is not too hard to show that weak equivalence implies flow equivalence under the appropriate conditions of which the most important is aperiodicity. And if two primitive substitutions are *starred* then the converse holds too:

Theorem 2. *Any two starred primitive substitutions are weakly equivalent if they are flow equivalent.*

That is, we have that two starred primitive substitutions are flow equivalent if and only if they are weakly equivalent and, as such, weak equivalence is a complete invariant for flow equivalence. For a substitution to be starred it must fulfill some rather harsh conditions regarding its fixed points, we can obtain this from any aperiodic primitive substitution by constructions that preserve of flow equivalence.

On the next slides we shall – somewhat loosely – build a weak equivalence.

Building a weak equivalence - Part I

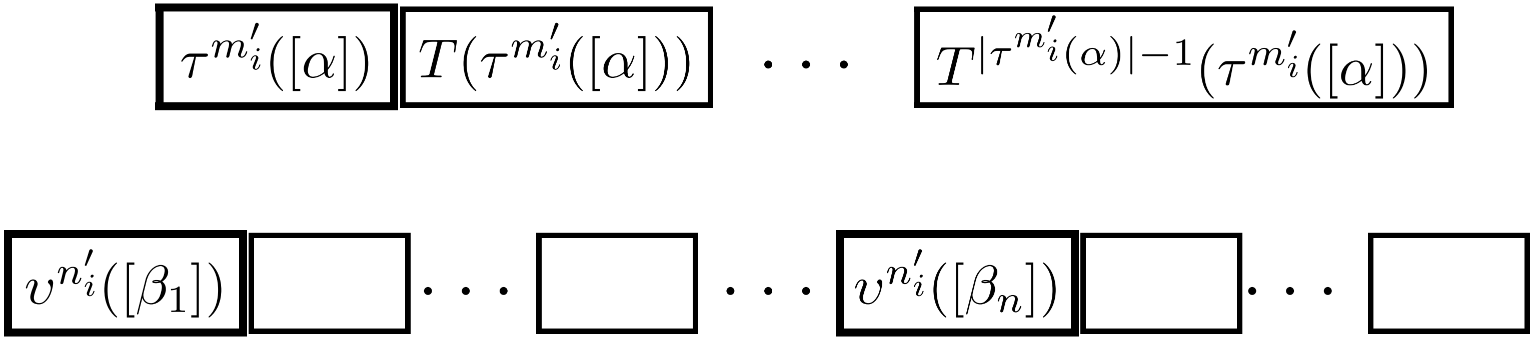
We construct two strictly increasing sequences of natural numbers $(m'_i)_{i\in\mathbb{N}}$ and $(n'_i)_{i\in\mathbb{N}}$, a sequence of maps $(\rho_i)_{i\in\mathbb{N}}$ from $\mathcal{A}$ to $\mathcal{B}^+$ and a sequence of maps $(\sigma_i)_{i\in\mathbb{N}}$ from $\mathcal{B}$ to $\mathcal{A}^+$. For $i \in \mathbb{N}$ we then prove commutativity of

$$\begin{tikzcd}
 & \mathcal{A}^* \arrow[dl, "\rho_i"'] & \\
\mathcal{B}^* & & \mathcal{B}^* \arrow[ul, "\sigma_i"'] \arrow[ll, "\upsilon^{n'_{i+1}-n'_i}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{A}^* & & \mathcal{A}^* \arrow[ll, "\tau^{m'_{i+1}-m'_i}"'] \arrow[dl, "\rho_{i+1}"'] \\
 & \mathcal{B}^* \arrow[ul, "\sigma_i"] &
\end{tikzcd}$$

Here we shall restrict ourselves to produce the leftmost map of the leftmost triangle. So assume $n'_i \in \mathbb{N}$ given, we are to choose m'_i and $\rho_i : \mathcal{A} \to \mathcal{B}^+$.

Building a weak equivalence - Part II

Take $\alpha \in \mathcal{A}$, we have to choose the value of $\rho_i(\alpha)$. We draw our way out:



$$\boxed{\tau^{m'_i}([\alpha])}\,\boxed{T(\tau^{m'_i}([\alpha]))}\quad \cdots \quad \boxed{T^{|\tau^{m'_i}(\alpha)|-1}(\tau^{m'_i}([\alpha]))}$$

$$\boxed{\upsilon^{n'_i}([\beta_1])}\,\boxed{\phantom{\upsilon^{n'_i}}}\cdots\,\boxed{\phantom{\upsilon^{n'_i}}}\;\cdots\;\boxed{\upsilon^{n'_i}([\beta_n])}\,\boxed{\phantom{\upsilon^{n'_i}}}\cdots\,\boxed{\phantom{\upsilon^{n'_i}}}$$

We need to make the top boxes small enough to fit in the bottom ones and we need to align the bold boxes properly. But both can be obtained by choosing m'_i sufficiently high – because the substitutions are starred.

Note that moving the top boxes to the bottom ones involves moving them across the entire flow equivalence and this introduces some technicalities.

Building a weak equivalence - Part III

But why do these constructions yield commutativity? For $\alpha \in \mathcal{A}$ we have

$$\boxed{\tau^{m'_{i+1}}([\alpha])}\boxed{T(\tau^{m'_{i+1}}([\alpha]))} \quad \cdots \quad \boxed{T^{|\tau^{m'_{i+1}}(\alpha)|-1}(\tau^{m'_{i+1}}([\alpha]))}$$

contained in

$$\boxed{\tau^{m'_i}([\alpha_1])}\boxed{\qquad\quad}\cdots\boxed{\qquad\quad}\cdots\boxed{\tau^{m'_i}([\alpha_n])}\boxed{\qquad\quad}\cdots\boxed{\qquad\quad}$$

where $\tau^{m'_{i+1}-m'_i}(\alpha) = \alpha_1\alpha_2\cdots\alpha_n$ for some $n \in \mathbb{N}$. This is true for all primitive substitutions. But as ρ_{i+1} and σ_i move boxes down and up in a uniform and inverse manner we also have the top boxes contained in

$$\boxed{\tau^{m'_i}([\alpha'_1])}\boxed{\qquad\quad}\cdots\boxed{\qquad\quad}\cdots\boxed{\tau^{m'_i}([\alpha'_{n'}])}\boxed{\qquad\quad}\cdots\boxed{\qquad\quad}$$

where $\sigma_i(\rho_{i+1}(\alpha)) = \alpha'_1\alpha'_2\cdots\alpha'_{n'}$ and we are done.