Logical- and Meta-Logical Frameworks
Lecture 5

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Recall from last time

Conclusion

- First order logic fits wonderfully into the framework.
- Importance of world checking.
- Proof by structural induction by cut elimination.
- Relies on coverage checking.
- Next time we will see how to conduct proofs by logical relations.

Proofs by logical relations
Going beyond inductive proofs
An application of cut-elimination.

Recall

The simply typed λ-calculus.

Types	$A, B ::= \mathrm{o} \mid A \rightarrow B$
Objects	$M, N ::= x \mid M\ N \mid \lambda x.\, M$
Contexts	$\Gamma ::= \cdot \mid \Gamma, x : A$
Remark	We only consider well-typed terms.

Recall from Lecture 2

Judgments

Canonical forms	$\Gamma \vdash M \Uparrow A$
Atomic forms	$\Gamma \vdash N \downarrow A$

Rules

$$\frac{x : A \in \Gamma}{\Gamma \vdash x \downarrow A} \qquad \frac{c : A \in \Sigma}{\Gamma \vdash c \downarrow A} \qquad \frac{\Gamma \vdash M \downarrow A \to B \quad \Gamma \vdash N \Uparrow A}{\Gamma \vdash M\ N \downarrow B}$$

$$\frac{\Gamma \vdash M \downarrow a}{\Gamma \vdash M \Uparrow a} \qquad \frac{\Gamma, x : A \vdash M \Uparrow B}{\Gamma \vdash \lambda x : A.\, M \Uparrow A \to B}$$

Judgments

- Oriented definitional equality: $M \longrightarrow M'$
- Oriented definitional equality: $M \longrightarrow^* M'$
- Existence of canonical objects: $\Gamma \vdash M \Uparrow M'$
- Existence of atomic objects: $\Gamma \vdash M \downarrow M'$

Conversion to Normal forms

$$\frac{}{(\lambda x.\, M)\, N \longrightarrow [N/x]M}\ \text{whr\_beta}$$

$$\frac{M \longrightarrow M'}{\Gamma \vdash M\ N \longrightarrow M'\ N}\ \text{whr\_cong}$$

. . . .

$$\frac{}{\Gamma \vdash x \downarrow x}\ \text{catm\_var} \qquad \frac{\Gamma \vdash M \downarrow M' \qquad \Gamma \vdash N \Uparrow N'}{\Gamma \vdash M\ N \downarrow M'\ N'}\ \text{catm\_app}$$

. . . .

$$\frac{\Gamma \vdash M \downarrow M'}{\Gamma \vdash M \Uparrow M'}\ \text{ccan\_atm} \qquad \frac{\Gamma \vdash M \longrightarrow M' \qquad \Gamma \vdash M' \Uparrow M''}{\Gamma \vdash M \Uparrow M''}\ \text{ccan\_whr}$$

$$\frac{\Gamma, x : A \vdash M\ x \Uparrow M'}{\Gamma \vdash M \Uparrow \lambda x : A.\, M'}\ \text{catm\_arr}$$

Goal

Theorem For every closed M of type A there exists an N of type A s.t. $M \longrightarrow^* N$ and $\vdash N \Uparrow$

Question How do we prove it?

Idea 1. Directly by induction? Doesn't work.

Idea 2. By a logical relations argument! Yes.

The logical relation

Definition Logical relation

$$
\begin{array}{lll}
\Gamma \vdash M \in [\![o]\!] & \text{iff} & \Gamma \vdash M \Uparrow N \text{ for some } N \\
\Gamma \vdash M \in [\![A \rightarrow B]\!] & \text{iff} & \text{for all } \Gamma' > \Gamma \\
 & & \text{and for all } \Gamma' \vdash N \in [\![A]\!] \\
 & & \text{implies } \Gamma \vdash M\ N \in [\![B]\!]
\end{array}
$$

Observation We use set theory as *assertion logic* to define the relation.

Meta theory

Fundamental Theorem

1. If $\Gamma \vdash M \in [\![A]\!]$ then $\Gamma \vdash M \Uparrow N$ for some N.
2. If $\Gamma \vdash M \downarrow$ then $\Gamma \vdash M \in [\![A]\!]$.

Weak-head reduction

If $\Gamma \vdash M \in [\![A]\!]$ and $M' \longrightarrow M$ then $\Gamma \vdash M' \in [\![A]\!]$.

Escape Theorem

If $\Gamma \vdash M \in [\![A]\!]$ then $\Gamma \vdash M \Uparrow N$ for some N.

Proof see blackboard.

And into LF

Formalize assertion logic: From yesterday.

Extend logic by new judgments.

- $\ulcorner \Gamma \vdash M \Uparrow M' \urcorner = \mathsf{hc}\ \ulcorner M \urcorner\ \ulcorner M' \urcorner : \mathsf{type}$
- $\ulcorner \Gamma \vdash M \downarrow M' \urcorner = \mathsf{ha}\ \ulcorner M \urcorner\ \ulcorner M' \urcorner : \mathsf{type}$
- $\ulcorner M \longrightarrow M' \urcorner = \mathsf{wh}\ \ulcorner M \urcorner\ \ulcorner M' \urcorner : \mathsf{type}$

Consistency of assertion logic.

Cut-elimination.

Twelf code

Let's look at it in Twelf.

Example Combinators

$$\begin{array}{lcl} K & : & \vdash A \supset B \supset A \\ S & : & \vdash (A \supset B \supset C) \supset (A \supset B) \supset (A \supset C) \end{array}$$

Interpret them as λ terms.

$$\begin{array}{lcl} K & = & \lambda x.\, \lambda y.\, x \\ S & = & \lambda x.\, \lambda y.\, \lambda z.\, (x\ z)\ (y\ z) \end{array}$$

Now we show that

$$S\ K\ K = \lambda x.\, x$$

Discussion

Assertion logic

Advantages	Modularity
	Scalability
	Consistency
Disadvantages	Depend on Twelf's expressive strength
	Weak compared with other meta logics.

Conclusion

Conclusion

- Judgments and Evidence.
- Types and objects in LF type theory.
- Terms are alive, substitution is built in.
- Twelf is a good tool to experiment with ideas.
- It will talk back to you.
- Meta-theory of deductive systems.
- Many practical applications.
  - Logosphere
  - POPLmark challenge
- Proofs by structural induction.
- Proofs by logical relations.