On the Adaptiveness of Quicksort

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Quicksort

- Introduced by Hoare in 1961
- Simple, randomized sorting algorithm
- Expected number of comparisons $\sim 1.4 n \log_2 n$ [Hoare'62]
- Expected number of swaps is $1/6$ the expected number of comparisons [Hoare'62]
- In-place sorting algorithm: elements are compared and swapped within the input array (plus a runtime stack)
- In practice very fast. The all-round sorting algorithm of choice (glibc, STL, JDK, .NET).

Adaptiveness

- Adaptive sorting - the running time depends both on the input size and the presortedness in the input
- A common measure of presortedness:

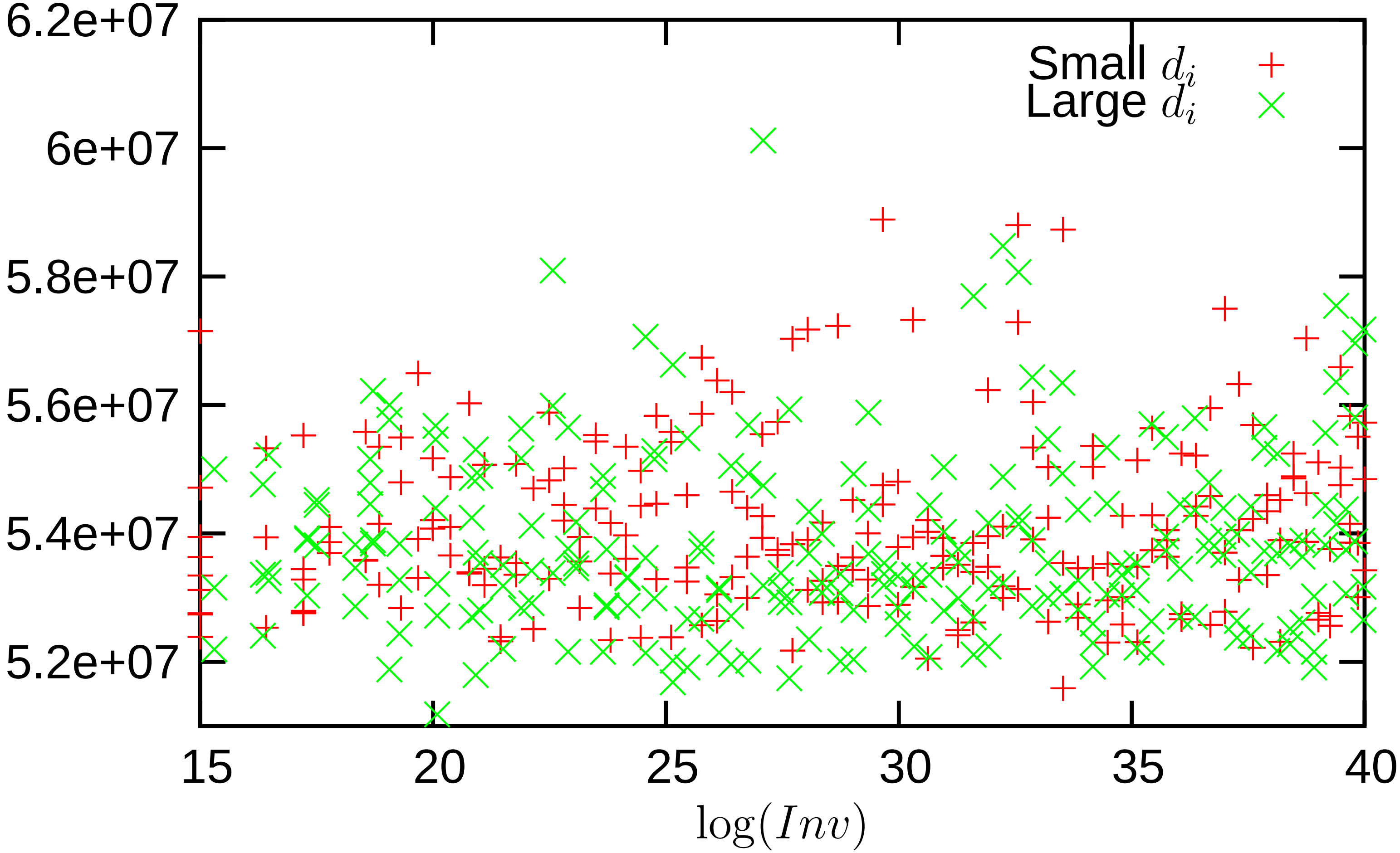
$$Inv(x_1 \dots x_n) = |\{(i,j) \mid i < j \wedge x_i > x_j\}|$$

$$Inv(1,2,3,4) = 0, Inv(4,3,2,1) = 6, Inv(2,1,4,3) = 2$$

- An optimal sorting algorithm with respect to Inv performs $\Theta(n(1 + \log(1 + \frac{Inv}{n})))$ comparisons [Manilla '85]

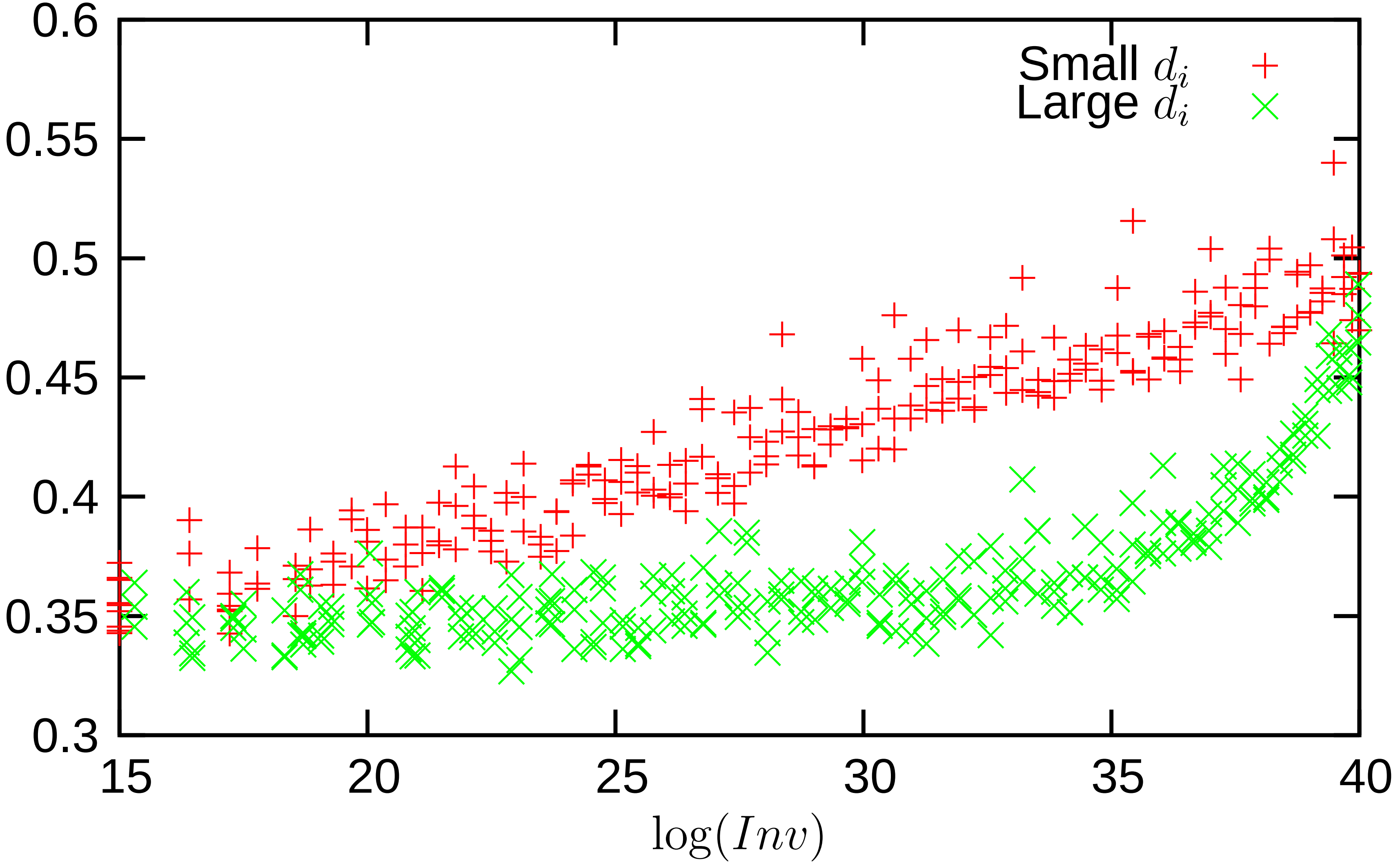
Quicksort (comparisons)

— Quicksort is not adaptive



Quicksort (running time)

— *Is Quicksort adaptive ?*



Results

Quicksort

- The number of comparisons is independent of the presortedness
- The number of swaps can be significantly smaller for nearly sorted inputs. We prove $O(n(1 + \log(1 + \frac{Inv}{n})))$.
- The number of branch mispredictions is given by the number of element swaps
- The running time is affected by more than a factor of two

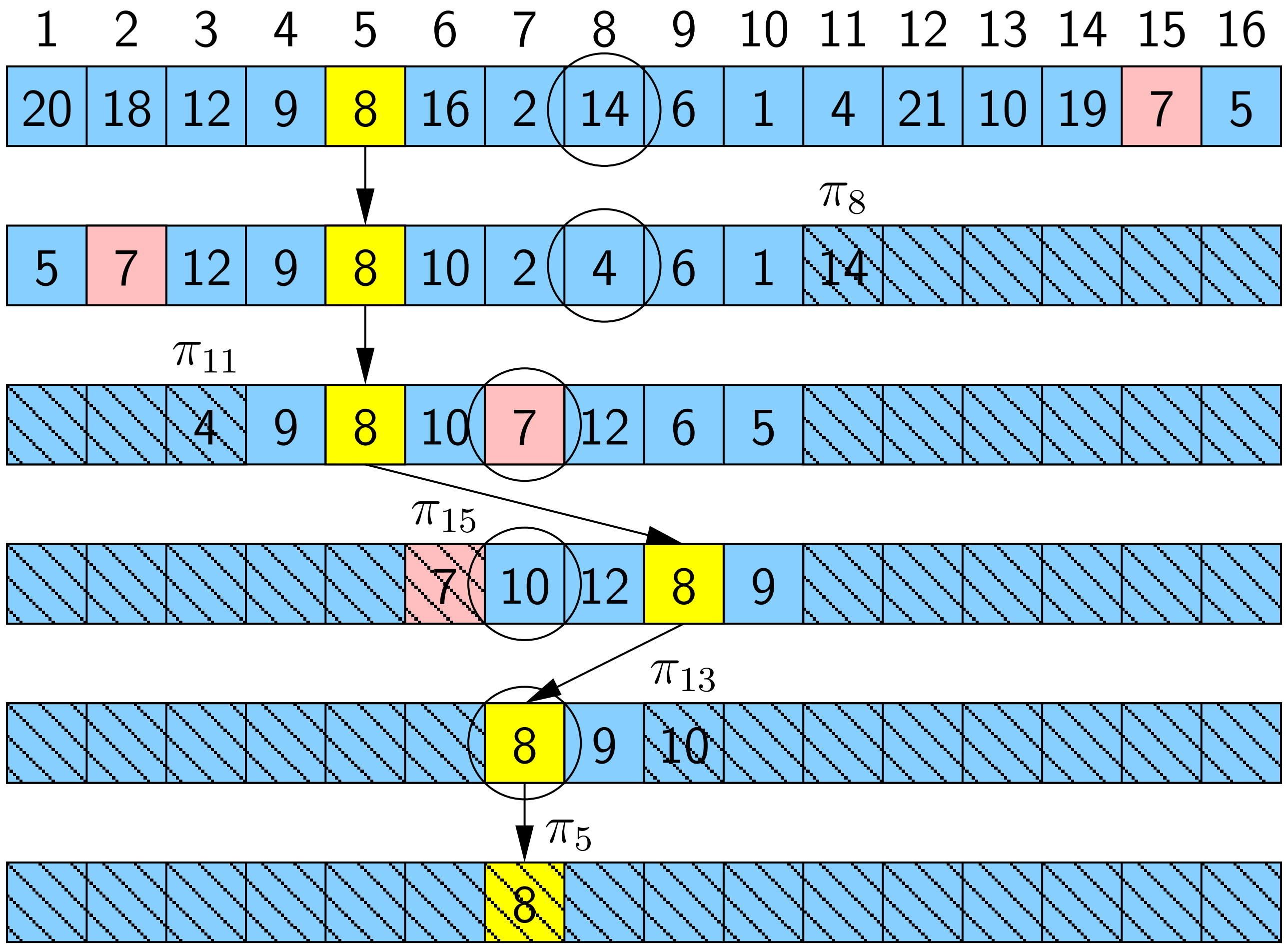
Binary Mergesort and Heapsort

- Empirical results are given

Quicksort C source

```
#define Item int
#define random(l,r) (l+rand() % (r-l+1))
#define swap(A, B) { Item t = A; A = B; B = t; }
void quicksort(Item a[], int l, int r)
   { int i;
     if (r <= l) return;
     i = partition(a, l, r);
     quicksort(a, l, i-1);
     quicksort(a, i+1, r);
   }
int partition(Item a[], int l, int r)
  { int i = l-1, j = r+1, p = random(l,r);
    Item v = a[p];
    for (;;) {
        while (++i < j && a[i] <= v);
        while (--j > i && v <= a[j]);
        if (j <= i) break;
        swap(a[i], a[j]);
      }
    if (p < i) i--;
    swap(a[i], a[p]);
    return i;
  }
```

Pivots vs Swaps



The first pivot causing $x_5 = 8$ to be swapped is $x_{15} = 7$

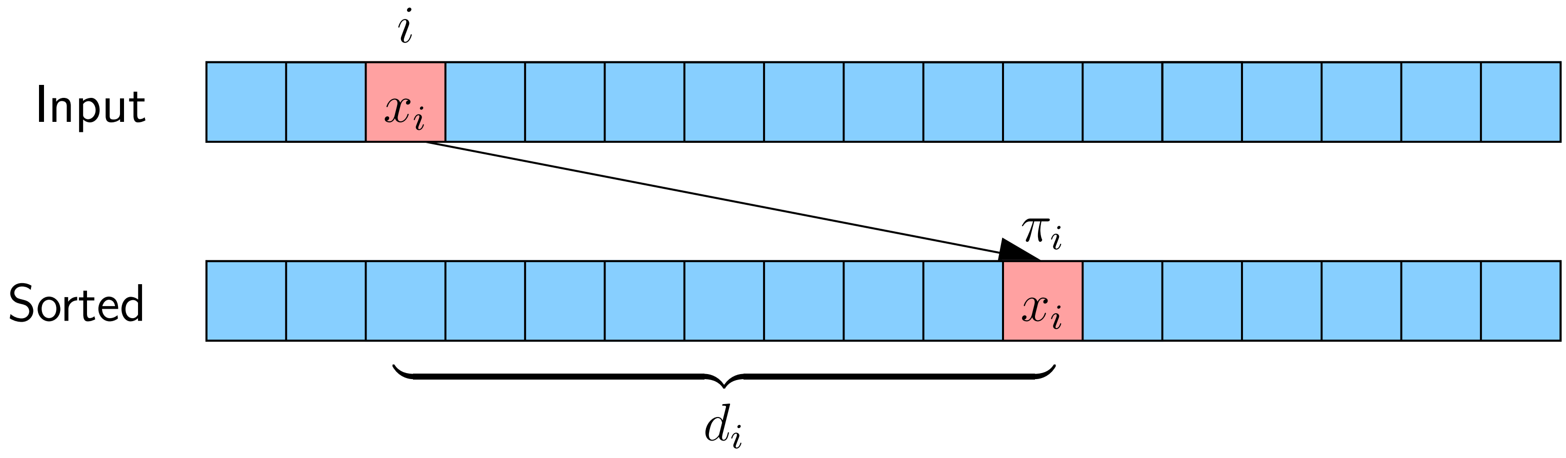
($\pi_5 = 7$, $\pi_{15} = 6$, and $5 \leq \pi_{15} < \pi_5$)

Main Theorem (I)

Theorem

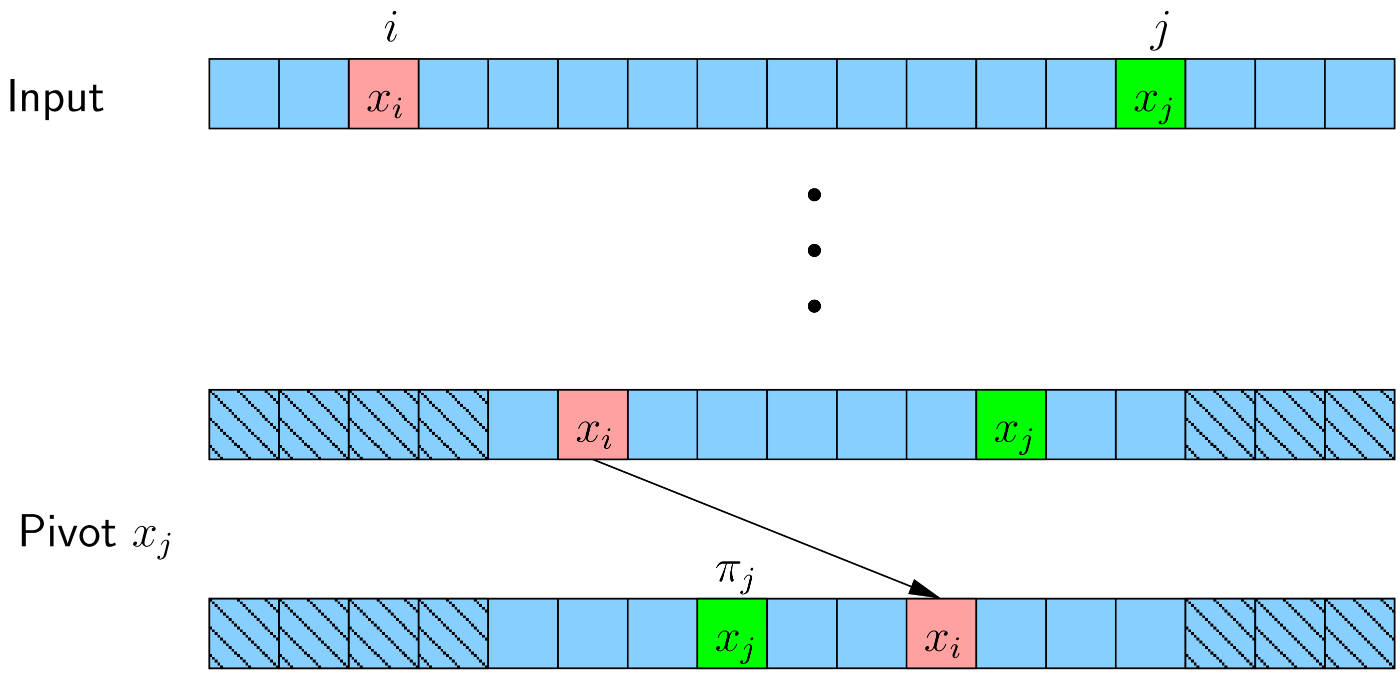
Quicksort performs expected $\leq n + n \ln \left(\frac{2Inv}{n} + 1 \right)$ swaps.

- $(x_1, \ldots, x_n)$ – input sequence of distinct elements
- π_i – rank of x_i in the sorted sequence
- $d_i = |\pi_i - i|$



Main Theorem (II)

Definition $X_{ij} = 1$ if when x_j becomes a pivot then x_i is swapped

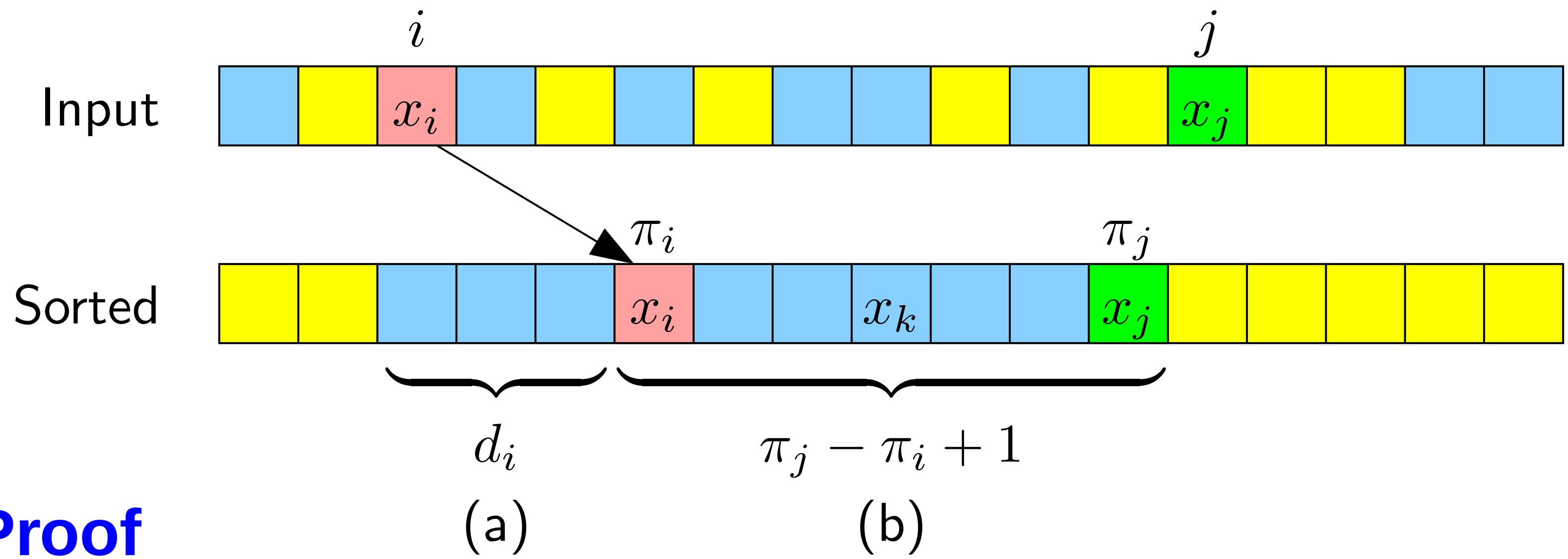


$$X_{ij} = 1$$

Main Theorem (III)

Lemma

$$\Pr[X_{ij}=1] \le \begin{cases} 0 & \text{if } \pi_j < i \le \pi_i \text{ or } \pi_i \le i < \pi_j \\ \frac{1}{|\pi_i-\pi_j|+1} & \text{if } i \le \pi_j < \pi_i \text{ or } \pi_i < \pi_j \le i \\ \frac{1}{|\pi_i-\pi_j|+1} - \frac{1}{|\pi_i-\pi_j|+1+d_i} & \text{otherwise} \end{cases}$$

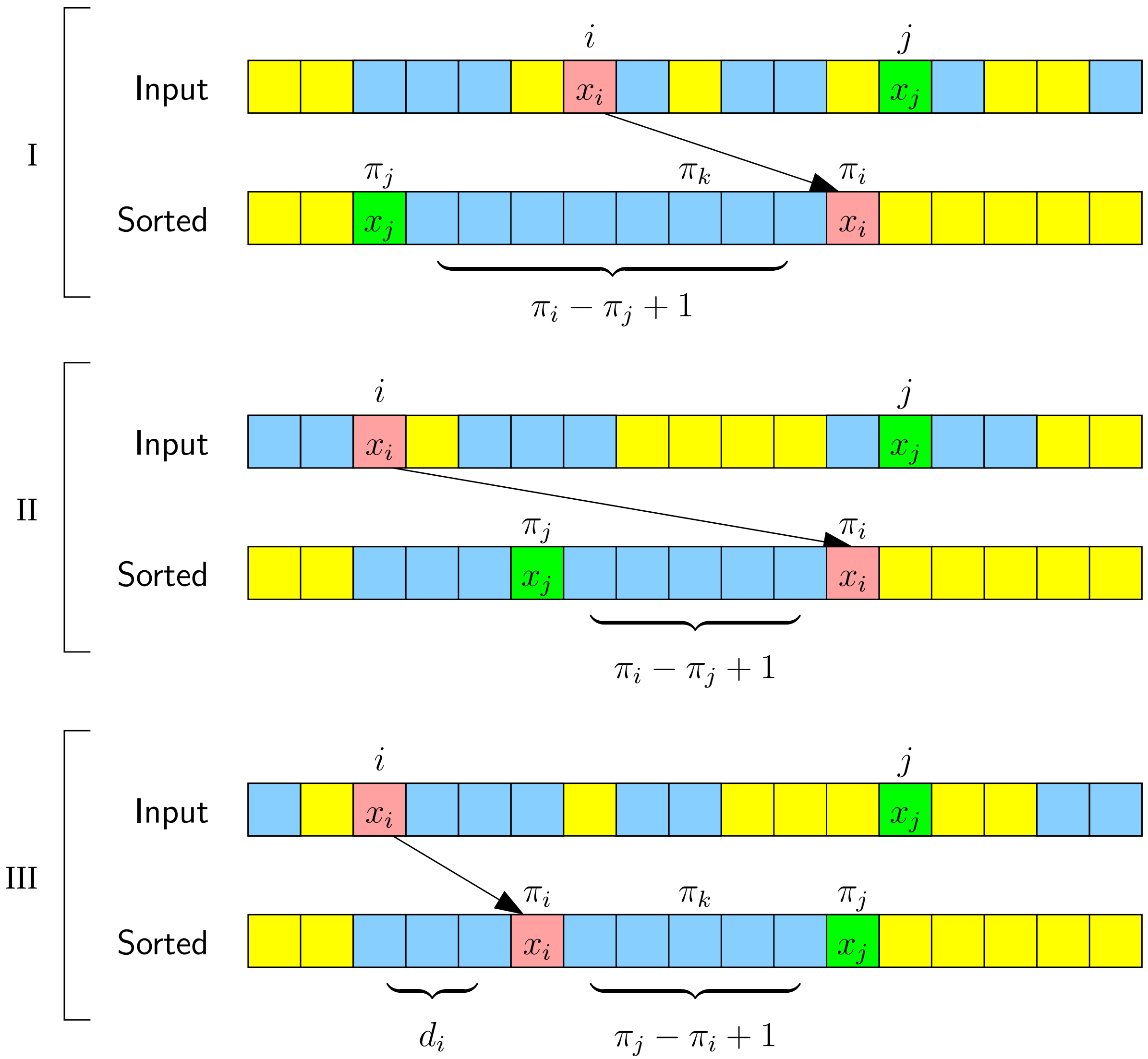


Proof

(a) Pivots forcing x_i to be swapped

(b) Pivots separating x_i and x_j

Main Theorem (IV)



$$Pr[X_{ij} = 1] = 0$$

$$Pr[X_{ij} = 1] \leq \frac{1}{|\pi_i - \pi_j| + 1}$$

$$Pr[X_{ij} = 1] \leq \frac{1}{|\pi_i - \pi_j| + 1} - \frac{1}{|\pi_i - \pi_j| + 1 + d_i}$$

Main Theorem (V)

Theorem

Quicksort performs expected $\leq n + n \ln\left(\frac{2Inv}{n} + 1\right)$ swaps.

Proof

$$\mathrm{E}\left[\sum_{j=1}^{n}\left(1 + \frac{1}{2}\sum_{i=1, i\neq j}^{n} X_{ij}\right)\right] = n + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1, i\neq j}^{n} \Pr(X_{ij} = 1)$$

$$\leq n + \frac{1}{2}\sum_{i=1}^{n}\left(\sum_{k=1}^{d_i}\frac{1}{k+1} + \sum_{k=1}^{n}\left(\frac{1}{k+1} - \frac{1}{k+1+d_i}\right)\right)$$

$$\leq \sum_{i=1}^{n}\sum_{k=1}^{d_i+1}\frac{1}{k} \leq n + n\ln\left(\frac{2Inv}{n} + 1\right)$$

using $\sum_{i=1}^{n} d_i \leq 2Inv$

Experimental Setup

- Two types of input

  (1) x_i uniformly at random in $[i-d..i+d]$ for increasing d, i.e. small d_i

  (2) $x_i = i$ except for some random i where x_i is randomly in $[0..n-1]$, i.e. large d_i

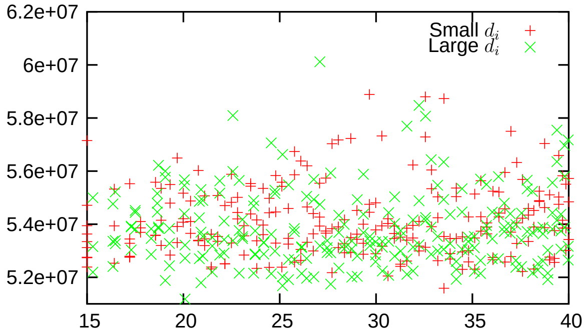
- Compare #comparisons, #swaps, #branch mispredictions, #L2 data cache misses and the running time against $\log(Inv)$

- $n = 2 \times 10^6$

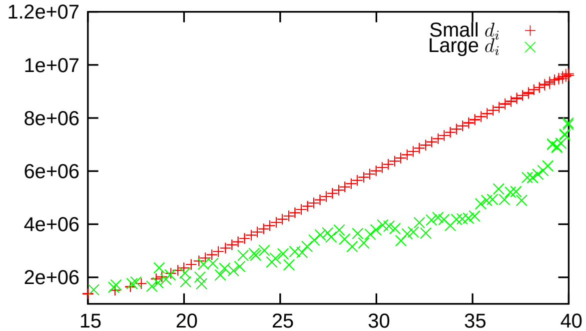
- AMD Athlon XP 2400+ 2.0 GHz, Redhat 9, Linux 2.4.20, gcc 3.3.2 using optimization -O3, PAPI 3.0

Experimental results (Quicksort)

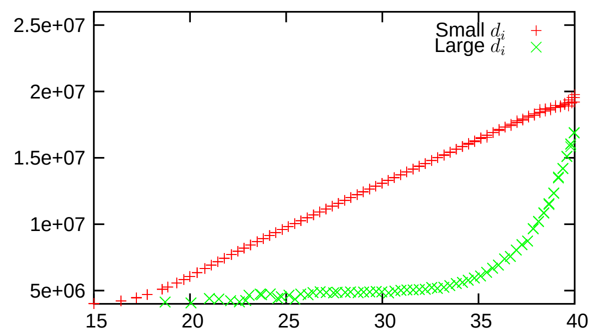
Comparisons
(10% difference)



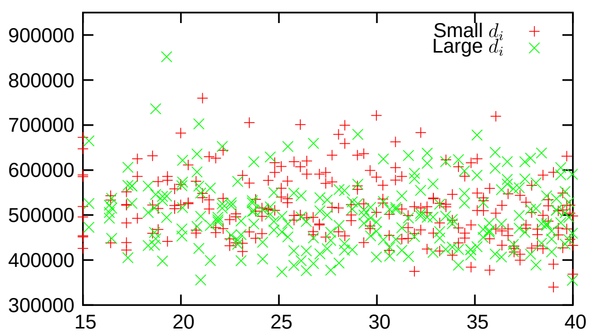
Swaps
(500% difference)



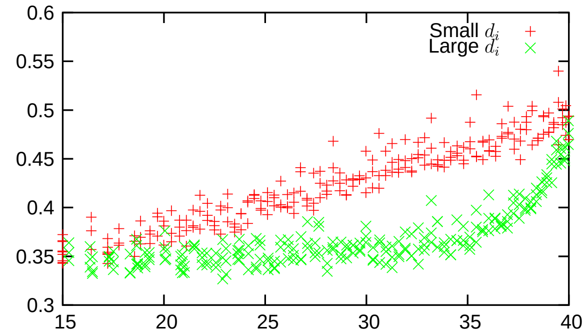
Mispredictions
(400%)



Cache misses
(60% difference)

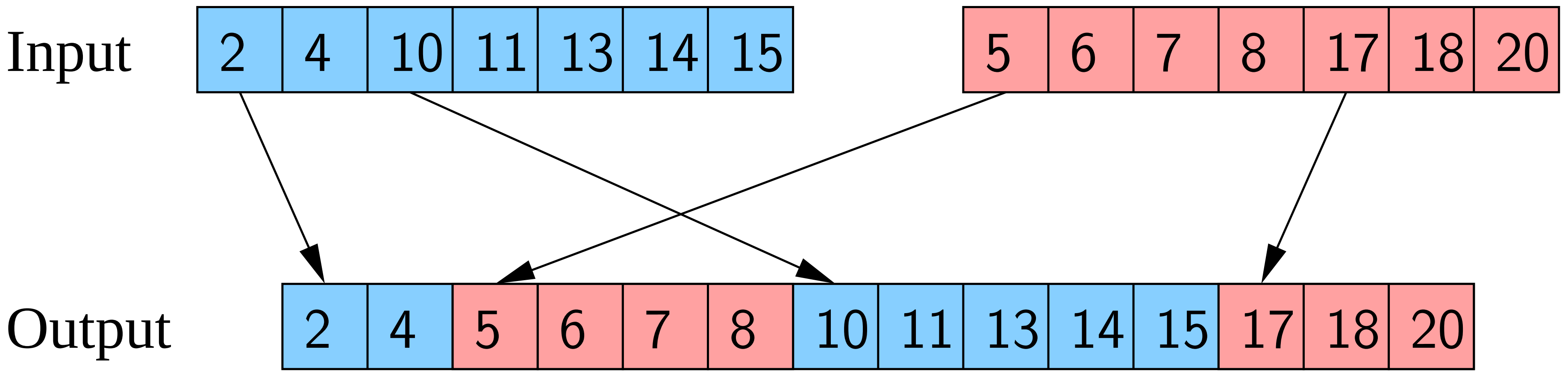


Running time
(50% difference)



Binary Mergesort

Alternations - the number of changes between the two input sequences in the result of a binary merging

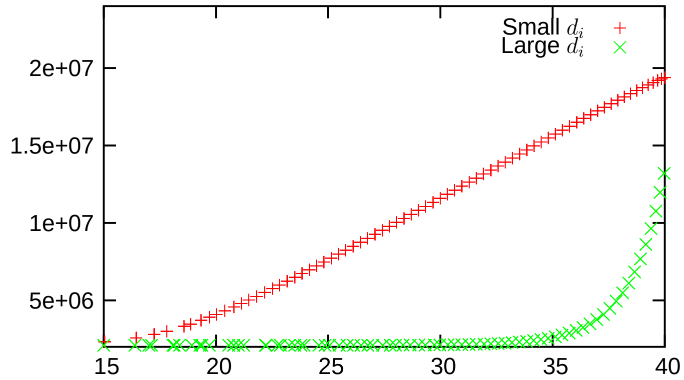


By result of Moffat et al.:

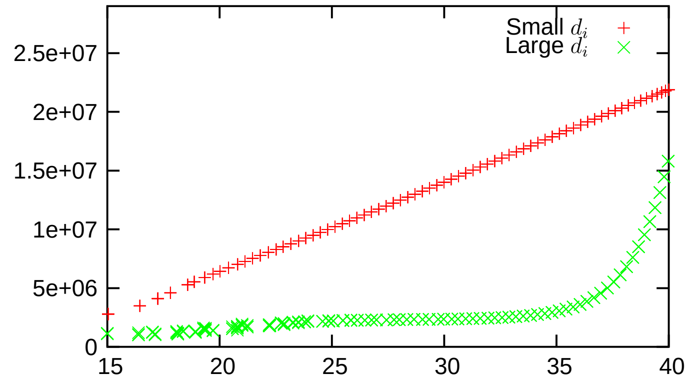
The number of alternations for Mergesort is $O(n \log \frac{Inv}{n})$

Experimental results (Mergesort)

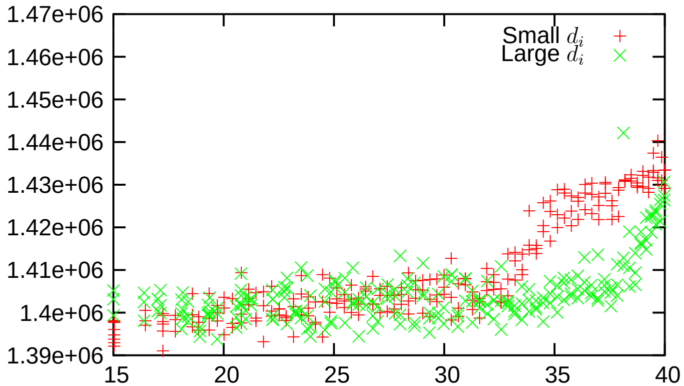
Alternations
(900% difference)



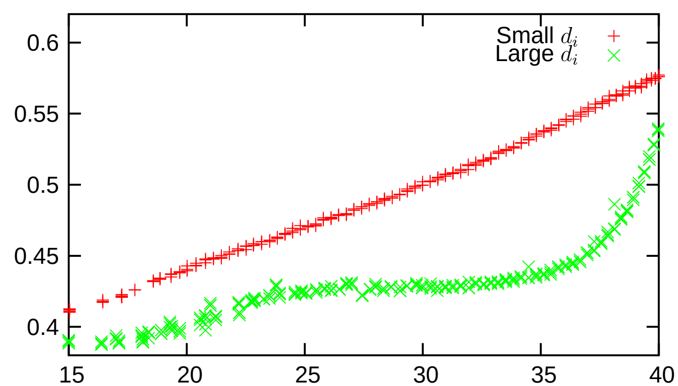
Mispredictions
(900% difference)



Cache misses
(5% difference)

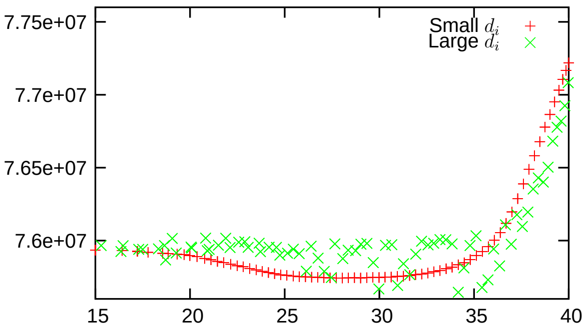


Running time
(35% difference)

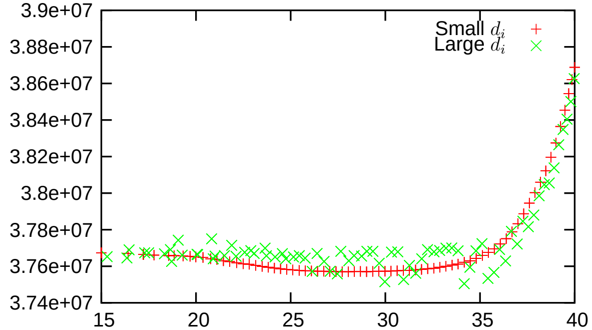


Experimental results (Heapsort)

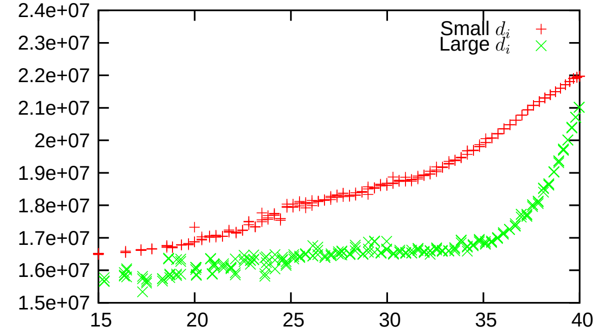
Comparisons
(5% difference)



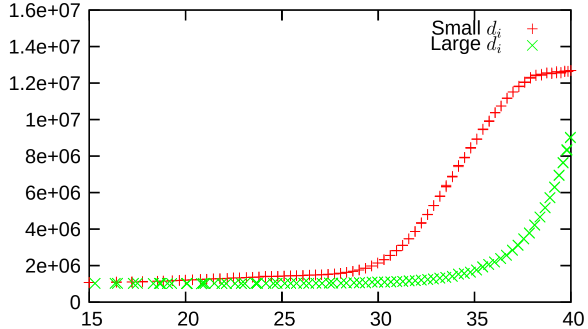
Swaps
(18% difference)



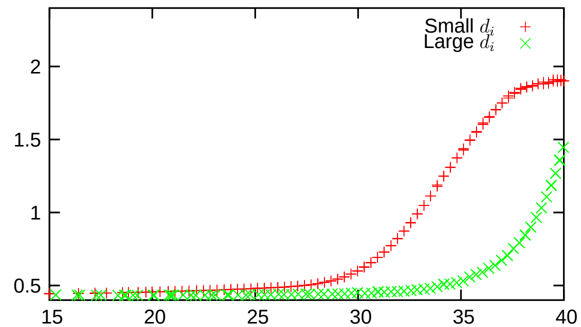
Mispredictions
(difference 30%)



Cache misses
(1000% difference)



Running time
(400% difference)



Conclusions

- Randomized Quicksort performs expected $O(n(1 + \log(1 + Inv/n)))$ swaps
- The number of branch mispredictions is given by the number of swaps
- The number of swaps performed can affect the running time of Quicksort by more than a factor of two
- Experimental results confirm the theoretical results for Quicksort
- Empirical results are given for Heapsort and Binary Mergesort