`n != 0 && -n == n`

`z+1 == z`

Computer arithmetics: integers, binary floating-point, and decimal floating-point

`v+w-w != v`

`y != y`

Peter Sestoft

2013-02-18**

`x+1 < x`

`p == n && 1/p != 1/n`

Computer arithmetics

- Computer numbers are cleverly designed, **but**
  - Very different from high-school mathematics
  - There are some surprises
- Choose representation with care:
  - When to use int, short, long, byte, …
  - When to use double or float
  - When to use decimal floating-point

Overview, number representations

- Integers
  - Unsigned, binary representation
  - Signed
    - Signed-magnitude
    - Two’s complement (Java and C# int, short, byte, …)
  - Arithmetic modulo 2^n
- Floating-point numbers
  - IEEE 754 binary32 and binary64
    - Which you know as `float` and `double` in Java and C#
  - IEEE 754 decimal128
    - and also C#’s `decimal` type
    - and also Java’s java.math.BigDecimal

Unsigned integers, binary representation

- Decimal notation

  $805_{10} = 8*10^2 + 0*10^1 + 5*10^0 = 805$

  Every place is worth 10 times that to the right

- Binary notation

  $1101_2 = 1*2^3 + 1*2^2 + 0*2^1 + 1*2^0 = 13$

  Every place is worth 2 times the right one

2^0	1
2^1	2
2^2	4
2^3	8
2^4	16
2^5	32
2^6	64
2^7	128
2^8	256

- Positional number systems:
  - Base is 10 or 2 or 16 or …
- Any non-positional number systems?

Binary numbers

- A bit is a binary digit: 0 or 1
- Easy to represent in electronics
- (But some base-10 hardware in the 1960es)
- Counting with three bits:

  000, 001, 010, 011, 100, 101, 110, 111
- Computing:

  1 + 1 = 10

  010 + 011 = 101

“There are 10 kinds of people: those who understand binary and those who don’t”

Hexadecimal numbers

- Hexidecimal numbers have base 16
- Digits: 0 1 2 3 4 5 6 7 8 9 A B C D E F

  $325_{16} = 3 * 16^2 + 2 * 16^1 + 5*16^0 = 805$

  Each place is worth 16 times that ...
- Useful alternative to binary
  - Because $16 = 2^4$
  - So 1 hex digit = 4 bits
- Computing in hex:

  A + B = 15

  AA + 1 = AB

  AA + 10 = BA

16^0	1
16^1	16
16^2	256
16^3	4096
16^4	65536

0	0000
1	0001
2	0010
3	0011
4	0100
5	0101
6	0110
7	0111

8	1000
9	1001
A	1010
B	1011
C	1100
D	1101
E	1110
F	1111

Negative integers

- Signed magnitude: A sign bit and a number
  - Problem: Then we have both +0 and -0
- Two's complement: Negate all bits and add 1
  - Only one zero
  - Easy to compute with
  - Requires known size of number, e.g. 8, 16, 32, 64 bits
- Examples of two's complement (4 bits)

-3 is represented by 1101 because 3 = 0011_2 so complement is 1100; add 1 to get -3 = 1101_2

-1 is represented by 1111 because 1 = 0001_2 so complement is 1110; add 1 to get -1 = 1111_2

-8 is represented by 1000 because 8 = 1000_2 so complement is 0111; add 1 to get -8 = 1000_2

Integer arithmetics modulo 2^n

- Java and C# **`int`** is 32-bit two's-complement
  - Max int is $2^{31}-1 = 2147483647$
  - Min int is $-(2^{31}) = -2147483648$
  - If $x = 2147483647$ then $x+1 = -2147483648 < x$
  - If $n = -2147483648$ then $-n = n$

```
00000000000000000000000000000000 = 0
00000000000000000000000000000001 = 1
00000000000000000000000000000010 = 2
00000000000000000000000000000011 = 3
01111111111111111111111111111111 = 2147483647
11111111111111111111111111111111 = -1
11111111111111111111111111111110 = -2
11111111111111111111111111111101 = -3
10000000000000000000000000000000 = -2147483648
```

An obviously non-terminating loop?

```
int i = 1;
while (i > 0)
  i++;
System.out.println(i);
```

Does terminate!

Values of i:
1
2
3
…
2147483646
2147483647
-2147483648

Binary fractions

- Before the point: …, 16, 8, 4, 2, 1
- After the point: 1/2, 1/4, 1/8, 1/16, …

$0.5 = 0.1_2$ $\qquad$ $2.125 = 10.001_2$

$0.25 = 0.01_2$ $\qquad$ $7.625 = 111.101_2$

$0.75 = 0.11_2$ $\qquad$ $118.625 = 1110110.101_2$

$0.125 = 0.001_2$

- But
  - how many digits are needed before the point?
  - how many digits are needed after the point?
- Answer: Binary floating-point (**`double`**, **`float`**)
  - The point is placed dynamically

Some nasty fractions

- Some numbers are not representable as finite decimal fractions:

  $1/7 = 0.\boxed{142857}142857142857\ldots_{10}$

- Same problem with binary fractions:

  $1/10 = 0.0001\boxed{1001}100110011001100\ldots_{2}$

- Quite unfortunate:
  - Float 0.10 is 0.100000001490116119384765625
  - So cannot represent 0.10 krone or $0.10 exactly
  - Nor 0.01 krone or $0.01 exactly
- **Do not** use binary floating-point (**`float`**, **`double`**) for accounting!

An obviously terminating loop?

```
double d = 0.0;
while (d != 1.0)
  d += 0.1;
```

Does **not** terminate!

Values of d:
0.1000000000000000000
0.2000000000000000000
0.3000000000000000004000
0.4000000000000000000
0.5000000000000000000
0.6000000000000000000
0.7000000000000000000
0.7999999999999999990000
0.8999999999999999990000
0.9999999999999999990000
1.0999999999999999990000
1.2000000000000000000
1.3000000000000000000

`d` never equals 1.0

History of floating-point numbers

- Until 1985: Many different designs, anarchy
  - Difficult to write portable (numerical) software
- Standard IEEE 754-1985 binary fp
  - Implemented by all modern hardware
  - Assumed by modern programming languages
  - Designed primarily by William Kahan for Intel
- Revised standard IEEE 754-2008
  - binary floating-point as in IEEE 754-1985
  - decimal floating-point (new)
- IEEE = "Eye-triple-E" = Institute of Electrical and Electronics Engineers (USA)

IEEE floating point representation

- Signed-magnitude
  - Sign, exponent, significand: $s * 2^{e-b} * c$
- Representation:
  - Sign s, exponent e, fraction f (= significand c minus 1)

```
s eeeeeeee fffffffffffffffffffffff
0 01111111 00000000000000000000000 = 1.0
```

`float`

Java, C#	bits	e bits	f bits	range	bias b	sign. digits
`float`, `binary32`	32	8	23	$\pm 10^{-44}$ to $\pm 10^{38}$	127	7
`double`, `binary64`	64	11	52	$\pm 10^{-323}$ to $\pm 10^{308}$	1023	15
Intel ext.	80	15	64	$\pm 10^{-4932}$ to $\pm 10^{4932}$	16635	19

Understanding the representation

- *Normalized* numbers
  - Choose exponent e so the significand is 1.ffffff…
  - Hence we need only store the .ffffff… not the 1.
- Exponent is unsigned but a bias is subtracted
  - For 32-bit float the bias b is 127

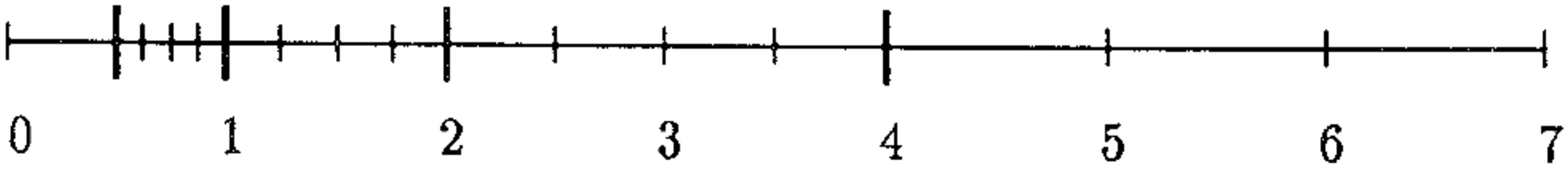
```
s eeeeeeee fffffffffffffffffffffff
0 00000000 00000000000000000000000 = 0.0
1 00000000 00000000000000000000000 = -0.0
0 01111111 00000000000000000000000 = 1.0
0 01111110 00000000000000000000000 = 0.5
1 10000101 11011010100000000000000 = -118.625
0 01111011 10011001100110011001101 = 0.1
0 01111111 00000000000000000000001 = 1.0000001
```

A detailed example

- Consider x = -118.625
- We know that 118.625 = 1110110.101_2
- Normalize to $2^6 * 1.110110101_2$
- So
  - exponent e = 6, represented by 6+127 = 133
  - significand is 1.110110101_2
  - so fraction f = $.110110101_2$
  - sign is 1 for negative

```
s eeeeeeee fffffffffffffffffffffff
1 10000101 11011010100000000000000 = -118.625
```

The normalized number line



- Representable with 2 f bits and 2 e bits:

  (So minimum e is -1 and maximum e is 2)

  $1.00_2 \times 2^{-1} = 0.5$
  $1.01_2 \times 2^{-1} = 0.625$
  $1.10_2 \times 2^{-1} = 0.75$
  $1.11_2 \times 2^{-1} = 0.875$
  $1.00_2 \times 2^{0} = 1$
  $1.01_2 \times 2^{0} = 1.25$
  $1.10_2 \times 2^{0} = 1.5$
  $1.11_2 \times 2^{0} = 1.75$

  $1.00_2 \times 2^{1} = 2$
  $1.01_2 \times 2^{1} = 2.5$
  $1.10_2 \times 2^{1} = 3$
  $1.11_2 \times 2^{1} = 3.5$
  $1.00_2 \times 2^{2} = 4$
  $1.01_2 \times 2^{2} = 5$
  $1.10_2 \times 2^{2} = 6$
  $1.11_2 \times 2^{2} = 7$

- Same relative precision for all numbers
- Decreasing absolute precision for large ones

Units in the last place (ulp)

- The distance between two neighbor numbers is called 1 ulp = unit in the last place

```
s eeeeeeee fffffffffffffffffffffff
0 01111111 00000000000000000000000 = 1.0
0 01111111 00000000000000000000001 = 1.0000001
```

1 ulp difference

- A good measure of
  - representation error
  - computation error
- Eg java.lang.Math.log documentation says

  "*The computed result must be within 1 ulp of the exact result.*"

Special "numbers"

- Denormal numbers, resulting from underflow
- Infinite numbers, resulting from 1.0/0.0, Math.log(0), …
- NaNs (not-a-number), resulting from 0.0/0.0, Math.sqrt(-1), …

Exponent e-b	Represented number
−126…127	Normal: $\pm 10^{-38}$ to $\pm 10^{38}$
−127	Denormal (underflow): $\pm 10^{-44}$ to $\pm 10^{-38}$, and ±0.0
128	Infinities, when f=0…0
128	NaNs, when f=1xx…xx

```
s eeeeeeee fffffffffffffffffffffff
1 10000101 11011010100000000000000 = -118.625
0 00000000 00010000000000000000000 = 7.346E-40
0 11111111 00000000000000000000000 = Infinity
1 11111111 00000000000000000000000 = -Infinity
s 11111111 10000000000000000000000 = NaN
```

Why denormal numbers?

- To allow gradual underflow, small numbers
- To ensure that x–y==0 if and only if x==y
- Example (32-bit float):
  - Smallest non-zero normal number is 2^{-126}
  - So choose $x=1.01_2*2^{-126}$ and $y=1.00_2*2^{-126}$:

```
s eeeeeeee fffffffffffffffffffffff
0 00000001 01000000000000000000000 = x
0 00000001 00000000000000000000000 = y
0 00000000 01000000000000000000000 = x-y
```

- What would happen without denormal?
  - Since x-y is 2^{-128} it is less than 2^{-126}
  - So result of x-y would be represented as 0.0
  - But clearly x!=y, so this would be confusing

Why infinities?

- 1: A simple solution to overflow
  - Math.exp(100000.0) gives +Infinity
- 2: To make "sensible" expressions work
  - Example: Compute $f(x) = x/(x^2+1.0)$
  - But if x is large then x^2 may overflow
  - Better compute: $f(x) = 1.0/(x+1.0/x)$
  - But if x=0 then 1.0/x looks bad, yet want f(0)=0
- Solution:
  - Let 1.0/0.0 be Infinity
  - Let 0.0+Infinity be Infinity
  - Let 1.0/Infinity be 0.0
  - Then 1.0/(0.0+1.0/0.0) gives 0 as should for x=0

Why NaNs?

- A simple and efficient way to report error
  - Languages like C do not have exceptions
  - Exceptions are 10,000 times slower than (1.2+x)
- Even weird expressions must have a result

  0.0/0.0 gives NaN

  Infinity – Infinity gives NaN

  Math.sqrt(-1.0) gives NaN

  Math.log(-1.0) gives NaN
- Operations must preserve NaNs

  NaN + 17.0 gives NaN

  Math.sqrt(NaN) gives NaN

  and so on

What about double (binary64)?

- The same, just with 64=1+11+52 bits instead of 32

```
s eeeeeeeeeee ffffffffffffffffffffffffffffffffffffffffffffffffffff
0 00000000000 0000000000000000000000000000000000000000000000000000 = 0.0
1 00000000000 0000000000000000000000000000000000000000000000000000 = -0.0
0 01111111111 0000000000000000000000000000000000000000000000000000 = 1.0
0 01111111110 0000000000000000000000000000000000000000000000000000 = 0.5
1 10000000101 1101101010000000000000000000000000000000000000000000 = -118.625
0 11111111111 0000000000000000000000000000000000000000000000000000 = Infinity
1 11111111111 0000000000000000000000000000000000000000000000000000 = -Infinity
s 11111111111 1000000000000000000000000000000000000000000000000000 = NaN
0 00000000000 0001000000000000000000000000000000000000000000000000 = 1.39E-309
1 11111111111 0000000000000000000000000000000000000000000000000000 = -Infinity
0 01111111011 1001100110011001100110011001100110011001100110011010 = 0.1
0 01111111110 1111111111111111111111111111111111111111111111111111 = 0.999...9
```

0.1+0.1+0.1+0.1+0.1+
0.1+0.1+0.1+0.1+0.1,
clearly not equal to 1.0

- Double 0.1 is really this exact number:
  0.1000000000000000055511151231257827021181583404541015625

IEEE addition

+	-Inf	-2.0	-0.0	0.0	2.0	+Inf	NaN
-Inf	-Inf	-Inf	-Inf	-Inf	-Inf	NaN	NaN
-2.0	-Inf	-4.0	-2.0	-2.0	0.0	+Inf	NaN
-0.0	-Inf	-2.0	-0.0	0.0	2.0	+Inf	NaN
0.0	-Inf	-2.0	0.0	0.0	2.0	+Inf	NaN
2.0	-Inf	0.0	2.0	2.0	4.0	+Inf	NaN
+Inf	NaN	+Inf	+Inf	+Inf	+Inf	+Inf	NaN
NaN	NaN	NaN	NaN	NaN	NaN	NaN	NaN

IEEE subtraction

-	-Inf	-2.0	-0.0	0.0	2.0	+Inf	NaN
-Inf	NaN	-Inf	-Inf	-Inf	-Inf	-Inf	NaN
-2.0	+Inf	0.0	-2.0	-2.0	-4.0	-Inf	NaN
-0.0	+Inf	2.0	0.0	-0.0	-2.0	-Inf	NaN
0.0	+Inf	2.0	0.0	0.0	-2.0	-Inf	NaN
2.0	+Inf	4.0	2.0	2.0	0.0	-Inf	NaN
+Inf	+Inf	+Inf	+Inf	+Inf	+Inf	NaN	NaN
NaN	NaN	NaN	NaN	NaN	NaN	NaN	NaN

IEEE multiplication

*	-Inf	-2.0	-0.0	0.0	2.0	+Inf	NaN
-Inf	+Inf	+Inf	NaN	NaN	-Inf	-Inf	NaN
-2.0	+Inf	4.0	0.0	-0.0	-4.0	-Inf	NaN
-0.0	NaN	0.0	0.0	-0.0	-0.0	NaN	NaN
0.0	NaN	-0.0	-0.0	0.0	0.0	NaN	NaN
2.0	-Inf	-4.0	-0.0	0.0	4.0	+Inf	NaN
+Inf	-Inf	-Inf	NaN	NaN	+Inf	+Inf	NaN
NaN	NaN	NaN	NaN	NaN	NaN	NaN	NaN

IEEE division

/	-Inf	-2.0	-0.0	0.0	2.0	+Inf	NaN
-Inf	NaN	+Inf	+Inf	-Inf	-Inf	NaN	NaN
-2.0	0.0	1.0	+Inf	-Inf	-1.0	-0.0	NaN
-0.0	0.0	0.0	NaN	NaN	-0.0	-0.0	NaN
0.0	-0.0	-0.0	NaN	NaN	0.0	0.0	NaN
2.0	-0.0	-1.0	-Inf	+Inf	1.0	0.0	NaN
+Inf	NaN	-Inf	-Inf	+Inf	+Inf	NaN	NaN
NaN	NaN	NaN	NaN	NaN	NaN	NaN	NaN

IEEE equality and ordering

==	-Inf	-2.0	-0.0	0.0	2.0	+Inf	NaN
-Inf	true	false	false	false	false	false	false
-2.0	false	true	false	false	false	false	false
-0.0	false	false	true	true	false	false	false
0.0	false	false	true	true	false	false	false
2.0	false	false	false	false	true	false	false
+Inf	false	false	false	false	false	true	false
NaN	false	false	false	false	false	false	false

- Equality (==, !=)
  - A NaN is not equal to anything, not even itself
  - So if y is NaN, then **`y != y`**
- Ordering: $-\infty < -2.0 < -0.0 == 0.0 < 2.0 < +\infty$
  - All ordering comparisons involving NaNs give false

Java and C# mathematical functions

- In general, functions behave sensibly
  - Give +Infinity or –Infinity on extreme arguments
  - Give NaN on invalid arguments
  - Preserve NaN arguments, with few exceptions

`sqrt(-2.0) = NaN`	`sqrt(NaN) = NaN`
`log(0.0) = -Inf`	`log(NaN) = NaN`
`log(-1.0) = NaN`	
`sin(Inf) = NaN`	`sin(NaN) = NaN`
`asin(2.0) = NaN`	
`exp(10000.0) = Inf`	`exp(NaN) = NaN`
`exp(-Inf) = 0.0`	
`pow(0.0, -1.0) = Inf`	`pow(NaN, 0.0) = 1 in Java`

Rounding modes

- High-school: round 0.5 upwards
  - Rounds 0,1,2,3,4 down and rounds 5,6,7,8,9 up
- Looks fair
- But dangerous: may introduce *drift* in loops

- IEEE-754:
  - Rounds 0,1,2,3,4 down and rounds 6,7,8,9 up
  - Rounds 0.5 to *nearest even* number (or more generally, to zero least significant bit)
- So both 1.5 and 2.5 round to 2.0

Basic principle of IEEE floating-point

"Each of the computational operations … shall be performed as if it first produced an intermediate result correct to infinite precision and unbounded range, and then rounded that intermediate result to fit in the destination's format"
(IEEE 754-2008 §5.1)

- So the machine result of x*y is the rounding of the "real" result of x*y
- This is simple and easy to reason about
- … and quite surprising that it can be implemented in finite hardware

Loss of precision 1 (ex: double)

- Let **`double`** $z=2^{53}$, then $z+1.0==z$
  - because only 52 digits in fraction

```
0 10000110100 0000000000000000000000000000000000000000000000000000=z
0 10000110100 0000000000000000000000000000000000000000000000000000=z+1
```

Loss of precision 2 (ex: double) Catastrophic cancellation

- Let v=9876543210.2 and w=9876543210.1
- Big and nearly equal; correct to 16 decimal places
- But their difference v–w is correct only to 6 places
- Because fractions were correct only to 6 places

```
v   = 9876543210.200000
w   = 9876543210.100000
v-w =            0.10000038146972656
```

Garbage, why?

```
v   = 9876543210.20000076293945312500
w   = 9876543210.10000038146972656250
v-w =            0.10000038146972656250
```

The exact actual numbers

```
0 10000100000 0010011001011000000001011011101010001100110011001101 0 = v
0 10000100000 0010011001011000000001011011101010000110011001100110 1 = w

0 01111111011 1001100110011010000000000000000000000000000000000000 = v-w
```

Would be non-zero in full-precision 0.1

Case: Solving a quadratic equation

- The solutions to $ax^2 + bx + c = 0$ are

$$x_1 = \frac{-b + \sqrt{d}}{2a} \qquad x_2 = \frac{-b - \sqrt{d}}{2a}$$

  when $d = b^2 - 4ac > 0$.
- But subtraction $-b \pm \sqrt{d}$ may lose precision when b^2 is much larger than $4ac$; in this case the square root is nearly b.
- Since $\sqrt{d} >= 0$, compute x_1 first if $b<0$, else compute x_2 first
- Then compute x_2 from x_1; or x_1 from x_2

Bad and good quadratic solutions

```
double d = b * b - 4 * a * c;
if (d > 0) {
  double y = Math.sqrt(d);
  double x1 = (-b - y)/(2 * a);
  double x2 = (-b + y)/(2 * a);
}
```

Bad

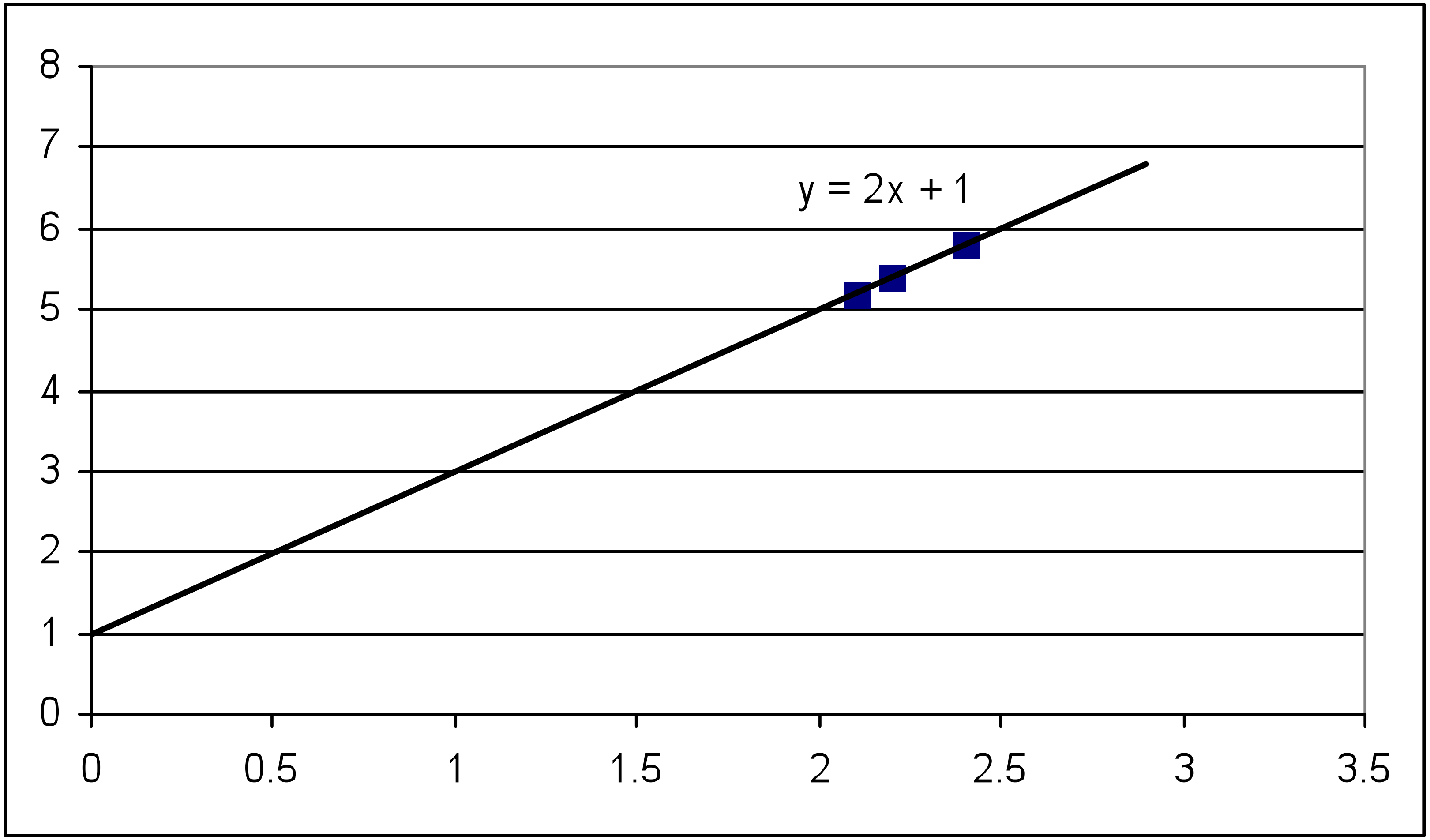
```
double d = b * b - 4 * a * c;
if (d > 0) {
  double y = Math.sqrt(d);
  double x1 = b > 0 ? (-b - y)/(2*a) : (-b + y)/(2*a);
  double x2 = c / (x1 * a);
} else ...
```

Good

- When a=1, b=10^9, c=1 we get
  - Bad algorithm: x1 = -1.00000e+09 and x2 = 0.00000
  - Good algorithm: x1 = -1.00000e+09 and x2 = -1.00000e-09

Case: Linear regression

- Points (2.1, 5.2), (2.2, 5.4), (2.4, 5.8) have regression line $y = \alpha + \beta x$ with $\alpha = 1$ and $\beta = 2$



Bad way to compute α and β

```
double SX = 0.0, SY = 0.0, SSX = 0.0, SXY = 0.0;
for (int i=0; i<n; i++) {
  Point p = ps[i];
  SX += p.x;
  SY += p.y;
  SXY += p.x * p.y;
  SSX += p.x * p.x;
}
double beta = (SXY - SX*SY/n) / (SSX - SX*SX/n);
double alpha = SY/n - SX/n * beta;
```

Large and nearly identical

- This recipe was used for computing by hand
- OK for points near (0,0)
- But otherwise may lose precision because it subtracts large numbers SSX and SX*SX/n

Better way to compute α and β

```
double SX = 0.0, SY = 0.0;
for (int i=0; i<n; i++) {
  Point p = ps[i];
  SX += p.x;
  SY += p.y;
}
double EX = SX/n, EY = SY/n;
double SDXDY = 0.0, SSDX = 0.0;
for (int i=0; i<n; i++) {
  Point p = ps[i];
  double dx = p.x - EX, dy = p.y - EY;
  SDXDY += dx * dy;
  SSDX += dx * dx;
}
double beta = SDXDY/SSDX;
double alpha = SY/n - SX/n * beta;
```

- Mathematically equivalent to previous one, but much more precise on the computer

Example results

- Consider (2.1, 5.2), (2.2, 5.4), (2.4, 5.8)
- And same with 10 000 000 or 50 000 000 added to each coordinate

Move		Bad	Good	Correct
0	α	1.000000	1.000000	**1.000000**
	β	2.000000	2.000000	**2.000000**
10 M	α	3.233333	-9999998.99	**-9999999.00**
	β	1.000000	2.000000	**2.000000**
50 M	α	50000005.47	-49999999.27	**-499999999.00**
	β	-0.000000	2.000000	**2.000000**

Wrong

Very wrong!!

An accurate computation of sums

20,000 elements

- Let `double[] xs = { 1E12, -1, 1E12, -1, … }`
- The true array sum is 9,999,999,999,990,000.0

```
double S = 0.0;
for (int i=0; i<xs.length; i++)
  S += xs[i];
```

Naïve sum, error = 992

```
double S = 0.0, C = 0.0;
for (int i=0; i<xs.length; i++) {
  double Y = xs[i] - C, T = S + Y;
  C = (T - S) - Y;
  S = T;
}
```

Kahan sum, error = 0

C is the error in the sum S

Note that C = (T-S)-Y = ((S+Y)-S)-Y may be non-zero

C# decimal, and IEEE decimal128

- C#'s **`decimal`** type is decimal floating-point
  - Has 28 significant digits
  - Has range $\pm 10^{-28}$ to $\pm 10^{28}$
  - Can represent 0.01 exactly
  - Uses 128 bits; computations are a little slower
- IEEE 754 **`decimal128`** is even better
  - Has 34 significant (decimal) digits
  - Has range $\pm 10^{-6143}$ to $\pm 10^{6144}$
  - Can represent 0.01 exactly
  - Uses 128 bits in a very clever way (Mike Cowlishaw, IBM)
- Java's java.math.BigDecimal
  - Has unlimited number of significant digits
  - Has range $\pm 10^{-2147483647}$ to $\pm 10^{2147483647}$
  - Computations are a lot slower

Use **decimal** for accounting (dollars, euro, kroner)!

Floating-point tips and tricks

- Do not compare floating-point using ==, !=
  - Use Math.abs(x−y) < 1E-9 or similar
  - Or better, compare difference in ulps (next slide)
- Do not use floating-point for currency ($, kr)
  - Use C# **`decimal`** or java.math.BigDecimal
  - Or use **`long`**, and store amount as cents or øre
- A **`double`** stores integers <= $2^{53}-1 \approx 8*10^{15}$ exactly
- To compute with very small positive numbers (probabilities) or very large positive numbers (combinations), use their logarithms

Approximate comparison

- Often useless to compare with "=="
- Fast relative comparison: difference in ulps
- Consider x and y as longs, subtract:

```
static boolean almostEquals(double x, double y, int maxUlps) {
  long xBits = Double.doubleToRawLongBits(x),
       yBits = Double.doubleToRawLongBits(y),
       MinValue = 1L << 63;
  if (xBits < 0)
    xBits = MinValue - xBits;
  if (yBits < 0)
    yBits = MinValue - yBits;
  long d = xBits - yBits;
  return d != MinValue && Math.abs(d) <= maxUlps;
}
```

```
1.0 == 0.1+0.1+0.1+0.1+0.1+0.1+0.1+0.1+0.1+0.1 is false
almostEquals(1.0, 0.1+0.1+0.1+0.1+0.1+0.1+0.1+0.1+0.1+0.1, 16) is true
```

What is that number *really?*

- Java's **`java.math.BigDecimal`** can display the *exact* number represented by double **`d`**:

```
new java.math.BigDecimal(d).toString()
```

```
double 0.125 = 0.125
float 0.125f = 0.125
```

```
double 0.1
       = 0.1000000000000000055511151231257827021181583404541015625
float 0.1f
       = 0.100000001490116119384765625
```

```
double 0.01
       = 0.01000000000000000020816681711721685132943093776702880859375
float 0.01f
       = 0.009999999776482582092285156 25
```

References

- David Goldberg: *What every computer scientist should know about floating-point arithmetics*, 1991. ACM Computing Surveys 23 (1) 1991.
  http://www.itu.dk/people/sestoft/bachelor/IEEE754_article.pdf
- R. Mak: *Java Number Cruncher: The Java Programmer's Guide to Numerical Computing.* Prentice-Hall 2002.
- Java example code and more:
  http://www.itu.dk/people/sestoft/bachelor/Numbers.java
  http://www.itu.dk/people/sestoft/javaprecisely/java-floatingpoint.pdf
  http://www.itu.dk/people/sestoft/papers/numericperformance.pdf
- http://en.wikipedia.org/wiki/IEEE_754-1985
- William Kahan notes on IEEE 754:
  http://www.cs.berkeley.edu/~wkahan/ieee754status/
  http://www.cs.berkeley.edu/~wkahan/ieee754status/754story.html
- General Decimal Arithmetic (Mike Cowlishaw, IBM)
  http://speleotrove.com/decimal/
- C# specification (Ecma International standard 334):
  http://www.ecma-international.org/publications/standards/Ecma-334.htm
- How to compare floating-point numbers (in C):
  http://www.cygnus-software.com/papers/comparingfloats/comparingfloats.htm