Solutions to the exercises on Sets, Relations, and Functions

April, 2007

Exercises on slide 11

Exercise 1

Argue that A and $\bar{A}$ are disjoint.

Solution

By definition of the complement, $\bar{A}$ is the set of those element from the universal set $\mathcal{U}$, which are not in A, so if $x \in A$ then $x \notin \bar{A}$ and if $x \in \bar{A}$ then $x \notin A$, thus there is no such x that $x \in A$ and $x \in \bar{A}$, therefore $A \cap \bar{A} = \emptyset$.

Exercise 2

Let $\mathcal{U} = \mathbb{N}$. What is the complement of $\{x : x^2 - 3x - 4 = 0\}$? What if $\mathcal{U} = \mathbb{Q}$?

Solution

$A = \{x : x^2 - 3x - 4 = 0\} = \{x : (x-4)(x+1) = 0\}$, then $\bar{A} = \{x : x^2 - 3x - 4 \neq 0\}$. Thus for $\mathcal{U} = \mathbb{N}$, $A = \{4\}$ and $\bar{A} = \{0, 1, 2, 3, 5, 6, 7, ...\}$. And for $\mathcal{U} = \mathbb{Q}$, $A = \{-1, 4\}$ and $\bar{A} = \mathbb{Q} \setminus \{-1, 4\}$.

Exercise on slide 12

Exercise 1

What's the power set of $\{a, b, c\}$?

Solution

$\mathcal{P}(\{a, b, c\}) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$

Exercise 2

Give an intuitive explanation of $\mathcal{P}(\mathbb{N})$.

Solution

$\mathcal{P}(\mathbb{N})$ is a set of all subset of $\mathbb{N}$ including the empty set $\emptyset$ and $\mathbb{N}$ itself.
$\mathcal{P}(\mathbb{N}) = \{\emptyset, \{0\}, \{1\}, \{2\}, ..., \{0,1\}, \{0,2\}, ...\{0,1,2\}, ...\{0,1,2,...,n,...\}, ..., \mathbb{N}\}$.

Exercises on slide 14

Exercise 1

Give a partition of the real numbers $\mathbb{R}$.

Solution

An example of a partion of $\mathbb{R}$ can be $\{A, B, C\}$, where $A = \{x : x > 0\}$, $B = \{0\}$ and $C = \{x : x < 0\}$. It is because A, B and C are not empty sets, they are pairwise disjoint and their union is equal to $\mathbb{R}$.

Exercise 2

Does there exist a partition of $\emptyset$?

Solution

The partition of $\emptyset$ is $\emptyset$.

The empty set can be written $\{A_i : i \in I\}$ where $I = \emptyset$. Recall the definition of a partition:
(a) $A_i \neq \emptyset$, for all $i \in I$
(b) $\bigcup_{i \in I} A_i = \emptyset$
(c) $A_i \cap A_j = \emptyset$, $i \neq j$, for all $i, j \in I$

(a) is trivially satisfied since $I = \emptyset$ from above. Also, (b) is vacously satisfied since the union of all sets indexed over by an empty set is empty. Finally, again since i and j range over an empty set there are no sets A_i and A_j so (c) holds trivially.

Exercise on slide 15

Give an example where $(a, b) \in A$ but $(b, a) \notin A$.

Solution

Let $A = \{(x, y) : x \text{ is a father of } y\}$. Then if Adam is a father of Bob, $(Adam, Bob) \in A$ but $(Bob, Adam) \notin A$, because Bob is a son of Adam, and so he cannot be his father.

Exercise on slide 21

Compute $(R_3 \circ R_2) \circ R_1$.

Solution

By the theorem on the lecture slide 21 $R_3 \circ (R_2 \circ R_1) = (R_3 \circ R_2) \circ R_1$.
Thus $(R_3 \circ R_2) \circ R_1 = \{(Adam, 30), (Bob, 63), (Chris, 52), (Dave, 30), (Eve, 63)\}$

Exercise on slide 22

Why is not $<$ on $\mathbb{N}$ an equivalence relation? Why is not $\leq$?

Solution

$<$ is not an equivalence relation on $\mathbb{N}$, because it is not reflexive. It follows from the fact that forall $a \in \mathbb{N}$ it holds that $a \not< a$.
$\leq$ is not an equivalence relation on $\mathbb{N}$, because it is not symmetric. A counter example illustrating that $\leq$ is not symmetric is $4 \leq 5$ but $5 \not\leq 4$.

Exercise on slide 27

Why is $\subseteq$ not a total order on $\mathcal{P}(A)$ if A contains at least two elements?

Solution

It is because not any two elements in $\mathcal{P}(A)$ can be related. For example, if $A = \{a, b\}$, then $\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. Here the two sets $\{a\}$ and $\{b\}$ cannot be related by $\subseteq$.

Exercise on slide 30

Let $f(x) = 2x + 3$ and $g(x) = 3x + 2$ be functions on $\mathbb{N}$. What is $(g \circ f)(x)$?

Solution

$(g \circ f)(x) = 3(f(x)) + 2 = 3(2x + 3) + 2 = 6x + 11$.

Exercises on slide 35

Exercise 2

Argue that for all $n \in \mathbb{Z}^+$ the relation R_n on $\mathbb{Z}^+$, defined by aR_nb if and only if $a\%n = b\%n$, is an equivalence relation.

Solution

In order to be an equivalence relation R_n must be reflexive(i), symmetric(ii) and transitive(iii).
(i) $\forall a \in \mathbb{Z}^+ : a\%n = a\%n$. Therefore R_n is reflexive.
(ii) $\forall a, b \in \mathbb{Z}^+ : a\%n = b\%n$ implies that $b\%n = a\%n$. Therefore R_n is symmetric.
(iii) $\forall a, b, c \in \mathbb{Z}^+ : a\%n = b\%n$ and $b\%n = c\%n$ implies that $a\%n = c\%n$. Therefore R_n is transitive.

Exercise 3

Argue why an equivalence relation that is also a function must be the identity. The **identity** $I : A \to A$ is defined by $I(a) = a$ for all $a \in A$.

Solution

Let R be an equivalence relation on A and a function $R : A \to A$. Then by the definition of a function, for every $a \in A$, there is one and only one $b \in A$ so that $(a, b) \in R$, which means that aRb. As it follows from the fact that an equivalence relation is reflexive, for every $a \in A$ aRa. Hence for every $a \in A$, there is one and only one $b \in A$ so that $(a, b) \in R$, and such $b = a$. Thus R is the identity.

Exercise 3.2.3 on page 80 in DM for NT

Let $\mathcal{U} = \{x : x \text{ is an integer and } 2 \leq x \leq 10\}$. In each of the following cases, determine whether $A \subseteq B$, $B \subseteq A$, both or neither:
(i) $A = \{x : x \text{ is odd }\}$ $B = \{x : x \text{ is a multiple of } 3\}$
(ii) $A = \{x : x \text{ is even}\}$ $B = \{x : x^2 \text{ is even}\}$
(iii) $A = \{x : x \text{ is even }\}$ $B = \{x : x \text{ is a power of } 2\}$
(iv) $A = \{x : 2x + 1 > 7\}$ $B = \{x : x^2 > 20\}$
(v) $A = \{x : \sqrt{x} \in \mathbb{Z}\}$ $B = \{x : x \text{ is a power of 2 or 3}\}$
(vi) $A = \{x : \sqrt{x} \leq 2\}$ $B = \{x : x \text{ is a perfect square}\}$
(vii) $A = \{x : x^2 - 3x + 2 = 0\}$ $B = \{x : x + 7 \text{ is a perfect square}\}$.

Solution

$\mathcal{U} = \{2, 3, 4, 5, 6, 7, 8, 9, 10\}$
(i) neither, $A = \{3, 5, 7, 9\}$ $B = \{3, 6, 9\}$
(ii) both, $A = \{2, 4, 6, 8, 10\}$ $B = \{2, 4, 6, 8, 10\}$
(iii) $B \subseteq A$, $A = \{2, 4, 6, 8, 10\}$ $B = \{2, 4, 8\}$
(iv) $B \subseteq A$, $A = \{4, 5, 6, 7, 8, 9, 10\}$ $B = \{5, 6, 7, 8, 9, 10\}$
(v) $A \subseteq B$, $A = \{4, 9\}$ $B = \{2, 3, 4, 8, 9\}$
(vi) neither, $A = \{2, 3, 4\}$ $B = \{4, 9\}$
(vii) both, $A = \{2\}$ $B = \{2\}$.

Exercise 3.2.8 on page 81 in DM for NT

(i) Prove that, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
(ii) Deduce that, if $A \subseteq B$, $B \subseteq C$ and $C \subseteq A$, then $A = B = C$.

Solution

(i) Let $x \in A$, then it follows from $A \subseteq B$ that $x \in B$, then it follows from $B \subseteq C$ that $x \in C$. This proves that every element of A also belongs to C, so $A \subseteq C$.
(ii) $A \subseteq B$, $B \subseteq C$, so it follows from (i) that $A \subseteq C$. If $A \subseteq C$ and $C \subseteq A$ then by the

theorem on the lecture slide 8 $A = C$. From $A = C$, $A \subseteq B$ and $B \subseteq C$ follows that $C \subseteq B$ and $B \subseteq C$, and thus by the same theorem $B = C$. Therefore $A = B = C$.

Exercise 3.2.10 on page 81 in DM for NT

Consider the set R of all sets which are not elements of themselves. That is, $R = \{A : A \text{ is a set and } A \notin A\}$.
Find a set which is an element of R. Can you find a set which is not an element of R?
Explain why R is not a well defined set. (Hint: is R itself an element of R?)

Solution

Let B is a set and $B \in R$, then by definition of R $B \notin B$. An example of such set B can be $B = \{1\}$ and many others, because usually $B \notin B$, like $\{1\} \notin \{1\}$.
Let us now find a set C which is not an element of R, so $C \in C$ must hold. An example of such set can be $C = \{\{...\{\{1\}\}...\}\}$, because $\{\{...\{\{1\}\}...\}\} \in \{\{\{...\{\{1\}\}...\}\}\}$.
R is not well defined, because assuming that $R \in R$, it follows from the definition of R that $R \notin R$, and assuming that $R \notin R$, it follows that $R \in R$. Such definition of R leads to a contradiction.

Exercise 3.5.4 on page 106 in DM for NT

Which of the following are partitions of $\mathbb{R}$, the set of real numbers? Explain your answers.
(i) $\{I_n : n \in \mathbb{Z}\}$, where $I_n = \{x \in \mathbb{R} : n \leq x \leq n + 1\}$.
(ii) $\{J_n : n \in \mathbb{Z}\}$, where $J_n = \{x \in \mathbb{R} : n \leq x < n + 1\}$.
(iii) $\{K_n : n \in \mathbb{Z}\}$, where $K_n = \{x \in \mathbb{R} : n < x < n + 1\}$.

Solution

(i) $\{I_n : n \in \mathbb{Z}\}$ is not a partition of $\mathbb{R}$, because $I_n \cap I_{n+1} \neq \emptyset$, and it follows from the fact that $\exists x = n + 1 : x \in I_n$ and $x \in I_{n+1}$.

(ii) $\{J_n : n \in \mathbb{Z}\}$ is a partition of $\mathbb{R}$, because $\forall n \in \mathbb{Z}\ J_n \neq \emptyset$, $\bigcup_{n\in\mathbb{Z}} J_n = \mathbb{R}$ and $J_i \cap J_j = \emptyset$ if $i \neq j$ for all $i, j \in \mathbb{Z}$

(iii) $\{K_n : n \in \mathbb{Z}\}$ is not a partition of $\mathbb{R}$, because $\bigcup_{n\in\mathbb{Z}} K_n \neq \mathbb{R}$, and it follows from the fact that $\forall i \in \mathbb{Z} : i \in \mathbb{R}$ and $i \notin K_n\ \forall n \in \mathbb{Z}$.

Exercise 5 on page 140 in DM with Combinatorics

Which of the following functions, whose domain and codomain are the real line, are one-to-one, which are onto, and which have inverses:
(a) $f(x) = |x|$
(b) $f(x) = x^2 + 4$
(c) $f(x) = x^3 + 6$

(d) $f(x) = x + |x|$
(e) $f(x) = x(x-2)(x+2)$

Solution

(a) $f(x) = |x|$.
This function is not one-to-one, because $\exists x_1$ and $\exists x_2 : f(x_1) = f(x_2)$ and $x_1 \neq x_2$, for example $x_1 = 3$ and $x_2 = -3$.
This function is not onto, because there exists such y, that for every $x : f(x) \neq y$, for example, $y < 0$, where $f(x) \geq 0$ for every x.
This function can have inverse $f^{-1}(y)$ only on $y \in [0, +\infty)$, because $f(x) \geq 0$ for all $x \in (-\infty, +\infty)$, moreover $f^{-1}(y) = \pm y$, that is not a function.

(b) $f(x) = x^2 + 4$
This function is not one-to-one, because $\exists x_1$ and $\exists x_2 : f(x_1) = f(x_2)$ and $x_1 \neq x_2$, for example $x_1 = 3$ and $x_2 = -3$.
This function is not onto, because there exists such y, that for every $x : f(x) \neq y$, for example, $y < 4$, where $f(x) \geq 4$ for every x.
This function can have inverse $f^{-1}(y)$ only on $y \in [4, +\infty)$, because $f(x) \geq 4$ for all $x \in (-\infty, +\infty)$, moreover $f^{-1}(y) = \pm\sqrt{y-4}$, that is not a function.

(c) $f(x) = x^3 + 6$
This function is one-to-one, because $\forall x_1$ and $\forall x_2 : f(x_1) = f(x_2)$ implies that $x_1 = x_2$, as it follows from $x_1^3 + 6 = x_2^3 + 6$ that $x_1 = x_2$.
This function is onto, because for every y, there exists $x : f(x) = y$.
This function have inverse $f^{-1}(y)$ on $y \in (-\infty, +\infty)$, and $f^{-1}(y) = \sqrt[3]{y-6}$, that is a function.

(d) $f(x) = x + |x|$
This function is not one-to-one, because $\exists x_1$ and $\exists x_2 : f(x_1) = f(x_2)$ and $x_1 \neq x_2$, for example $x_1 = -3$ and $x_2 = -4$.
This function is not onto, because there exists such y, that for every $x : f(x) \neq y$, for example, $y < 0$, where $f(x) \geq 0$ for every x.
This function can have inverse $f^{-1}(y)$ only on $y \in [0, +\infty)$, because $f(x) \geq 0$ for all $x \in (-\infty, +\infty)$, moreover for $y > 0$ the inverse is defined as $f^{-1}(y) = \frac{y}{2}$, and for $y = 0$ $f^{-1}(y) = a$, where a can be any real number that ≤ 0, so such inverse is not a function.

(e) $f(x) = x(x-2)(x+2)$
This function is not one-to-one, because $\exists x_1$ and $\exists x_2 : f(x_1) = f(x_2)$ and $x_1 \neq x_2$, for example $x_1 = 0$ and $x_2 = 2$.
This function is onto, because for every y, there exists $x : f(x) = y$.
This function have inverse $f^{-1}(y)$ on $y \in (-\infty, +\infty)$, and $f^{-1}(0) = 0, 2$ or -2, that is not a function.

Exercise 5 on page 160 in DM with Combinatorics

Show that the set $A = \{-10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, ...\}$ is countably infinite.

Solution

This set A is countably infinite, because there exists a bijection $f : A \to \mathbb{Z}^+$, where $f(a) = a + 11$ for $a \in A$.