Exercises on slide 11

Exercise 1
Argue that $A$ and $\bar{A}$ are disjoint.

Solution
By definition of the complement, $\bar{A}$ is the set of those elements from the universal set $\mathbb{U}$, which are not in $A$, so if $x \in A$ then $x \notin \bar{A}$ and if $x \in \bar{A}$ then $x \notin A$, thus there is no such $x$ that $x \in A$ and $x \in \bar{A}$, therefore $A \cap \bar{A} = \emptyset$.

Exercise 2
Let $\mathbb{U} = \mathbb{N}$. What is the complement of $\{x : x^2 - 3x - 4 = 0\}$? What if $\mathbb{U} = \mathbb{Q}$?

Solution
$A = \{x : x^2 - 3x - 4 = 0\} = \{x : (x - 4)(x + 1) = 0\}$, then $\bar{A} = \{x : x^2 - 3x - 4 \neq 0\}$. Thus for $\mathbb{U} = \mathbb{N}$, $A = \{4\}$ and $\bar{A} = \{0, 1, 2, 3, 5, 6, 7, \ldots\}$. And for $\mathbb{U} = \mathbb{Q}$, $A = \{-1, 4\}$ and $\bar{A} = \mathbb{Q} \setminus \{-1, 4\}$.

Exercise on slide 12

Exercise 1
What’s the power set of $\{a, b, c\}$?

Solution
$\mathcal{P}(\{a, b, c\}) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$

Exercise 2
Give an intuitive explanation of $\mathcal{P}(\mathbb{N})$. 
Solution

\( \mathcal{P}(\mathbb{N}) \) is a set of all subset of \( \mathbb{N} \) including the empty set \( \emptyset \) and \( \mathbb{N} \) itself.

\[ \mathcal{P}(\mathbb{N}) = \{ \emptyset, \{0\}, \{1\}, \{2\}, \ldots, \{0, 1\}, \{0, 2\}, \ldots, \{0, 1, 2\}, \ldots, \{0, 1, 2, \ldots, n, \ldots\}, \ldots, \mathbb{N} \}. \]

Exercises on slide 14

Exercise 1

Give a partition of the real numbers \( \mathbb{R} \).

Solution

An example of a partition of \( \mathbb{R} \) can be \( \{A, B, C\} \), where \( A = \{x : x > 0\} \), \( B = \{0\} \) and \( C = \{x : x < 0\} \). It is because \( A \), \( B \) and \( C \) are not empty sets, they are pairwise disjoint and their union is equal to \( \mathbb{R} \).

Exercise 2

Does there exist a partition of \( \emptyset \)?

Solution

The partition of \( \emptyset \) is \( \emptyset \).

The empty set can be written \( \{A_i : i \in I\} \) where \( I = \emptyset \). Recall the definition of a partition:

(a) \( A_i \neq \emptyset \), for all \( i \in I \)
(b) \( \bigcup_{i \in I} A_i = \emptyset \)
(c) \( A_i \cap A_j = \emptyset \), \( i \neq j \), for all \( i, j \in I \)

(a) is trivially satisfied since \( I = \emptyset \) from above. Also, (b) is vacuously satisfied since the union of all sets indexed over an empty set is empty. Finally, again since \( i \) and \( j \) range over an empty set there are no sets \( A_i \) and \( A_j \) so (c) holds trivially.

Exercise on slide 15

Give an example where \( (a, b) \in A \) but \( (b, a) \notin A \).

Solution

Let \( A = \{(x, y) : x \text{ is a father of } y\} \). Then if Adam is a father of Bob, \( (Adam, Bob) \in A \) but \( (Bob, Adam) \notin A \), because Bob is a son of Adam, and so he cannot be his father.

Exercise on slide 21

Compute \((R_3 \circ R_2) \circ R_1\).
Solution

By the theorem on the lecture slide 21 \( R_3 \circ (R_2 \circ R_1) = (R_3 \circ R_2) \circ R_1 \).
Thus \( (R_3 \circ R_2) \circ R_1 = \{(Adam, 30), (Bob, 63), (Chris, 52), (Dave, 30), (Eve, 63)\} \)

Exercise on slide 22

Why is not \(<\) on \(\mathbb{N}\) an equivalence relation? Why is not \(\leq\)?

Solution

\(<\) is not an equivalence relation on \(\mathbb{N}\), because it is not reflexive. It follows from the fact that for all \(a \in \mathbb{N}\) it holds that \(a \not< a\).
\(\leq\) is not an equivalence relation on \(\mathbb{N}\), because it is not symmetric. A counter example illustrating that \(\leq\) is not symmetric is \(4 \leq 5\) but \(5 \not\leq 4\).

Exercise on slide 27

Why is \(\subseteq\) not a total order on \(\mathcal{P}(A)\) if \(A\) contains at least two elements?

Solution

It is because not any two elements in \(\mathcal{P}(A)\) can be related. For example, if \(A = \{a, b\}\), then \(\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}\). Here the two sets \(\{a\}\) and \(\{b\}\) cannot be related by \(\subseteq\).

Exercise on slide 30

Let \(f(x) = 2x + 3\) and \(g(x) = 3x + 2\) be functions on \(\mathbb{N}\). What is \((g \circ f)(x)\)?

Solution

\((g \circ f)(x) = 3(f(x)) + 2 = 3(2x + 3) + 2 = 6x + 11\).

Exercises on slide 35

Exercise 2

Argue that for all \(n \in \mathbb{Z}^+\) the relation \(R_n\) on \(\mathbb{Z}^+\), defined by \(aR_nb\) if and only if \(a\%n = b\%n\), is an equivalence relation.

Solution

In order to be an equivalence relation \(R_n\) must be reflexive(i), symmetric(ii) and transitive(iii).
(i) \(\forall a \in \mathbb{Z}^+: a\%n = a\%n\). Therefore \(R_n\) is reflexive.
(ii) \(\forall a, b \in \mathbb{Z}^+: a\%n = b\%n\) implies that \(b\%n = a\%n\). Therefore \(R_n\) is symmetric.
(iii) \(\forall a, b, c \in \mathbb{Z}^+: a\%n = b\%n\) and \(b\%n = c\%n\) implies that \(a\%n = c\%n\). Therefore \(R_n\) is transitive.
Exercise 3
Argue why an equivalence relation that is also a function must be the identity. The identity $I : A \to A$ is defined by $I(a) = a$ for all $a \in A$.

Solution
Let $R$ be an equivalence relation on $A$ and a function $R : A \to A$. Then by the definition of a function, for every $a \in A$, there is one and only one $b \in A$ so that $(a, b) \in R$, which means that $aRb$. As it follows from the fact that an equivalence relation is reflexive, for every $a \in A$ $aRa$. Hence for every $a \in A$, there is one and only one $b \in A$ so that $(a, b) \in R$, and such $b = a$. Thus $R$ is the identity.

Exercise 3.2.3 on page 80 in DM for NT
Let $U = \{x : x$ is an integer and $2 \leq x \leq 10\}$. In each of the following cases, determine whether $A \subseteq B$, $B \subseteq A$, both or neither:
(i) $A = \{x : x$ is odd $\}$ $B = \{x : x$ is a multiple of 3 $\}$
(ii) $A = \{x : x$ is even $\}$ $B = \{x : x^2$ is even $\}$
(iii) $A = \{x : x$ is even $\}$ $B = \{x : x$ is a power of 2 $\}$
(iv) $A = \{x : 2x + 1 > 7\} B = \{x : x^2 > 20\}$
(v) $A = \{x : \sqrt{x} \in \mathbb{Z}\} B = \{x : x$ is a power of 2 or 3 $\}$
(vi) $A = \{x : \sqrt{x} \leq 2\} B = \{x : x$ is a perfect square $\}$
(vii) $A = \{x : x^2 - 3x + 2 = 0\} B = \{x : x + 7$ is a perfect square $\}$.

Solution
$U = \{2, 3, 4, 5, 6, 7, 8, 9, 10\}$
(i) neither, $A = \{3, 5, 7, 9\} B = \{3, 6, 9\}$
(ii) both, $A = \{2, 4, 6, 8, 10\} B = \{2, 4, 6, 8, 10\}$
(iii) $B \subseteq A$, $A = \{2, 4, 6, 8, 10\} B = \{2, 4, 8\}$
(iv) $B \subseteq A$, $A = \{4, 5, 6, 7, 8, 9, 10\} B = \{5, 6, 7, 8, 9, 10\}$
(v) $A \subseteq B$, $A = \{4, 9\} B = \{2, 3, 4, 8, 9\}$
(vi) neither, $A = \{2, 3, 4\} B = \{4, 9\}$
(vii) both, $A = \{2\} B = \{2\}$.

Exercise 3.2.8 on page 81 in DM for NT
(i) Prove that, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
(ii) Deduce that, if $A \subseteq B$, $B \subseteq C$ and $C \subseteq A$, then $A = B = C$.

Solution
(i) Let $x \in A$, then it follows from $A \subseteq B$ that $x \in B$, then it follows from $B \subseteq C$ that $x \in C$. This proves that every element of $A$ also belongs to $C$, so $A \subseteq C$.
(ii) $A \subseteq B$, $B \subseteq C$, so it follows from (i) that $A \subseteq C$. If $A \subseteq C$ and $C \subseteq A$ then by the
theorem on the lecture slide 8 $A = C$. From $A = C$, $A \subseteq B$ and $B \subseteq C$ follows that $C \subseteq B$ and $B \subseteq C$, and thus by the same theorem $B = C$. Therefore $A = B = C$.

**Exercise 3.2.10 on page 81 in DM for NT**

Consider the set $R$ of all sets which are not elements of themselves. That is, $R = \{ A : A \text{ is a set and } A \notin A \}$.

Find a set which is an element of $R$. Can you find a set which is not an element of $R$?

Explain why $R$ is not a well defined set. (Hint: is $R$ itself an element of $R$?)

**Solution**

Let $B$ is a set and $B \in R$, then by definition of $R$ $B \notin B$. An example of such set $B$ can be $B = \{1\}$ and many others, because usually $B \notin B$, like $\{1\} \notin \{1\}$.

Let us now find a set $C$ which is not an element of $R$, so $C \notin C$ must hold. An example of such set can be $C = \{\ldots\{\{1\}\}\ldots\}$, because $\{\ldots\{\{1\}\}\ldots\} \in \{\ldots\{\{1\}\}\ldots\}$.

$R$ is not well defined, because assuming that $R \in R$, it follows from the definition of $R$ that $R \notin R$, and assuming that $R \notin R$, it follows that $R \in R$. Such definition of $R$ leads to a contradiction.

**Exercise 3.5.4 on page 106 in DM for NT**

Which of the following are partitions of $\mathbb{R}$, the set of real numbers? Explain your answers.

(i) $\{I_n : n \in \mathbb{Z}\}$, where $I_n = \{ x \in \mathbb{R} : n \leq x \leq n + 1 \}$.

(ii) $\{J_n : n \in \mathbb{Z}\}$, where $J_n = \{ x \in \mathbb{R} : n \leq x < n + 1 \}$.

(iii) $\{K_n : n \in \mathbb{Z}\}$, where $K_n = \{ x \in \mathbb{R} : n < x < n + 1 \}$.

**Solution**

(i) $\{I_n : n \in \mathbb{Z}\}$ is not a partition of $\mathbb{R}$, because $I_n \cap I_{n+1} \neq \emptyset$, and it follows from the fact that $\exists x = n + 1 : x \in I_n$ and $x \in I_{n+1}$.

(ii) $\{J_n : n \in \mathbb{Z}\}$ is a partition of $\mathbb{R}$, because $\forall n \in \mathbb{Z} J_n \neq \emptyset$, $\bigcup_{n \in \mathbb{Z}} J_n = \mathbb{R}$ and $J_i \cap J_j = \emptyset$ if $i \neq j$ for all $i, j \in \mathbb{Z}$.

(iii) $\{K_n : n \in \mathbb{Z}\}$ is not a partition of $\mathbb{R}$, because $\bigcup_{n \in \mathbb{Z}} K_n \neq \mathbb{R}$, and it follows from the fact that $\forall i \in \mathbb{Z} : i \in \mathbb{R}$ and $i \notin K_n \forall n \in \mathbb{Z}$.

**Exercise 5 on page 140 in DM with Combinatorics**

Which of the following functions, whose domain and codomain are the real line, are one-to-one, which are onto, and which have inverses:

(a) $f(x) = |x|

(b) $f(x) = x^2 + 4

(c) $f(x) = x^3 + 6
(d) \( f(x) = x + |x| \)
(e) \( f(x) = x(x - 2)(x + 2) \)

Solution
(a) \( f(x) = |x| \).
This function is not one-to-one, because \( \exists x_1 \) and \( \exists x_2 : f(x_1) = f(x_2) \) and \( x_1 \neq x_2 \), for example \( x_1 = 3 \) and \( x_2 = -3 \).
This function is not onto, because there exists such \( y \), that for every \( x : f(x) \neq y \), for example, \( y < 0 \), where \( f(x) \geq 0 \) for every \( x \).
This function can have inverse \( f^{-1}(y) \) only on \( y \in [0, +\infty) \), because \( f(x) \geq 0 \) for all \( x \in (-\infty, +\infty) \), moreover \( f^{-1}(y) = \pm y \), that is not a function.

(b) \( f(x) = x^2 + 4 \)
This function is not one-to-one, because \( \exists x_1 \) and \( \exists x_2 : f(x_1) = f(x_2) \) and \( x_1 \neq x_2 \), for example \( x_1 = 3 \) and \( x_2 = -3 \).
This function is not onto, because there exists such \( y \), that for every \( x : f(x) \neq y \), for example, \( y < 4 \), where \( f(x) \geq 4 \) for every \( x \).
This function can have inverse \( f^{-1}(y) \) only on \( y \in [4, +\infty) \), because \( f(x) \geq 4 \) for all \( x \in (-\infty, +\infty) \), moreover \( f^{-1}(y) = \pm \sqrt{y - 4} \), that is not a function.

(c) \( f(x) = x^3 + 6 \)
This function is one-to-one, because \( \forall x_1 \) and \( \forall x_2 : f(x_1) = f(x_2) \) implies that \( x_1 = x_2 \), as it follows from \( x_1^3 + 6 = x_2^3 + 6 \) that \( x_1 = x_2 \).
This function is onto, because for every \( y \), there exists \( x : f(x) = y \).
This function have inverse \( f^{-1}(y) \) on \( y \in (-\infty, +\infty) \), and \( f^{-1}(y) = \sqrt[3]{y - 6} \), that is a function.

(d) \( f(x) = x + |x| \)
This function is not one-to-one, because \( \exists x_1 \) and \( \exists x_2 : f(x_1) = f(x_2) \) and \( x_1 \neq x_2 \), for example \( x_1 = -3 \) and \( x_2 = -4 \).
This function is not onto, because there exists such \( y \), that for every \( x : f(x) \neq y \), for example, \( y < 0 \), where \( f(x) \geq 0 \) for every \( x \).
This function can have inverse \( f^{-1}(y) \) only on \( y \in [0, +\infty) \), because \( f(x) \geq 0 \) for all \( x \in (-\infty, +\infty) \), moreover for \( y > 0 \) the inverse is defined as \( f^{-1}(y) = \frac{y}{2} \), and for \( y = 0 \ f^{-1}(y) = a \), where \( a \) can be any real number that \( \leq 0 \), so such inverse is not a function.

(e) \( f(x) = x(x - 2)(x + 2) \)
This function is not one-to-one, because \( \exists x_1 \) and \( \exists x_2 : f(x_1) = f(x_2) \) and \( x_1 \neq x_2 \), for example \( x_1 = 0 \) and \( x_2 = 2 \).
This function is onto, because for every \( y \), there exists \( x : f(x) = y \).
This function have inverse \( f^{-1}(y) \) on \( y \in (-\infty, +\infty) \), and \( f^{-1}(0) = 0, 2 \) or \(-2 \), that is not a function.
Exercise 5 on page 160 in DM with Combinatorics

Show that the set \( A = \{-10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, \ldots\} \) is countably infinite.

Solution

This set \( A \) is countably infinite, because there exists a bijection \( f : A \to \mathbb{Z}^+ \), where \( f(a) = a + 11 \) for \( a \in A \).