Approximate Range Emptiness in Constant Time and Optimal Space

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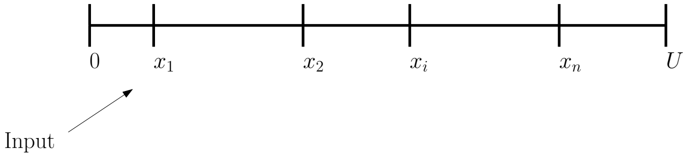

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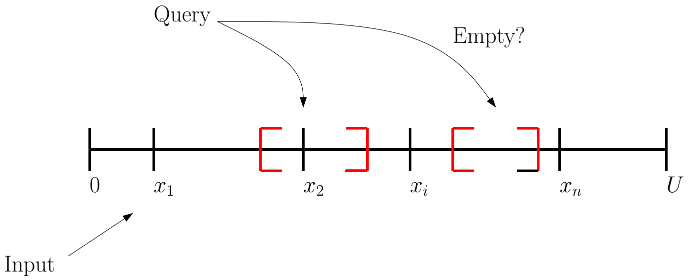

- Input a set S of n elements from $[U]$. Preprocess it to answer
- Query: $[a, b]$; is $[a, b] \cap S \neq \emptyset$?

Motivation: Exact versus Approximate Membership

- **Membership:** Given a set $S = \{x_1, \cdots, x_n\}$ from a universe $[U]$, preprocess the set to answer membership queries for a queried element q ($q \in S$?).

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- Reduction in space if we only want ϵ-approximate answers?
- Yes. **Bloom Filters**[1] $O(n \lg(1/\epsilon))$ space, $O(k)$ query. FPR ϵ.
- Here k is the number of hash functions used, and depends on ϵ.
- Optimal Bloom Filters (Pagh et. al.): Query time $O(1)$ irrespective of ϵ and space usage $(1 + o(1))n \lg(1/\epsilon)$.

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Approximate Range Emptiness

Range queries are more frequent in real life than membership queries.

- **Range emptiness:** Minimum space required $B = \lg \binom{U}{n}$ bits. Follows from membership.
- Alstrup et. al.: $O(n)$ words $= O(n \lg U)$ bits, $O(k)$ reporting, where k is the number of reported points.
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- **Approximate range emptiness (ARE):** False negatives not allowed. A fraction ϵ of false positives allowed.
- Of all the $u^2/2$ range queries, only an ϵ fraction may have false positives.

Main Question

Can we reduce space usage for range queries to something lower than $n \lg U$, by requiring approximate answers, similar to membership versus approximate membership queries?

One way to do ARE

- Let us say we want a data structure that answers only to ranges of size at most $L < U$
- One way to do approx. range emptiness query on $[a, b]$ is to
  - Build a Bloom Filter on S with FPR ϵ/L.
  - For every $x \in [a, b]$, run a membership query on the Bloom Filter.
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Results: Lower Bounds

Lower Bounds

We first show that the space error tradeoff cannot be improved significantly.

Theorem

Any data structure for the ARE problem answering all query intervals of a fixed length $L \leq u/5n$ with false positive rate $\varepsilon > 0$, must use at least

$$s \geq n \lg \left(\frac{L^{1-O(\varepsilon)}}{\varepsilon} \right) - O(n)$$

bits of space.

Extension to Two Sided Errors

Theorem

Any data structure for ARE with two sided error rate ϵ must use

$$
\begin{aligned}
s &\geq n \lg(L/\varepsilon) - O(n) && \textit{bits when } 0 < \varepsilon < 1/\lg U, \\
s &= \Omega\left(\frac{n \lg(L \lg U)}{\lg_{1/\varepsilon} \lg U}\right) && \textit{bits when } \frac{1}{\lg U} \leq \varepsilon \leq \frac{1}{2} - \Omega(1)
\end{aligned}
$$

Results: Upper Bounds

Upper Bounds

- There is a data structure $\mathcal{D}_a$ for the ARE problem that
  - answers range emptiness for all ranges of length at most L,
  - uses $n \lg(L/\varepsilon) + O(n \lg^{\delta}(L/\varepsilon))$ bits of space, δ any desired constant, and
  - has a false positive probability at most ϵ.

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  - has a false positive probability at most ϵ.
- A data structure $\mathcal{D}_e$ that
  - uses $n \lg(U/n) + o(n \lg^{\delta} U/n)$ bits[2],
  - answers exact range reporting in $O(k)$ and exact emptiness in $O(1)$ time, respectively.

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  2. Pairwise independence: $\Pr[g(x) = y, g(x^{'}) = y^{'}] = 1/R^2$ for all $x \neq x^{'}$ in $[U/R]$ and all $y, y^{'}$ in $[R]$.

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  3. Define $f(x) = (g(\lfloor x/R \rfloor) + x) \bmod R$.

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- If either is non-empty in $f(S)$ we report non-empty, else report empty.
- No false negatives. False positives occur when $x \in S$ and $y \in [a, b]$ collide.

$$\sum_{x \in S} \sum_{y \in I} \Pr[f(x) = f(y)] \leq nL/r \leq \epsilon.$$

$\mathcal{D}_e$: The ERE Data Structure

- First D_e^*: Store n elts. from $[U]$ in $n \lg U + O(n \lg^\delta U)$ bits, $\delta > 0$ any desired constant, and answer queries in constant time.
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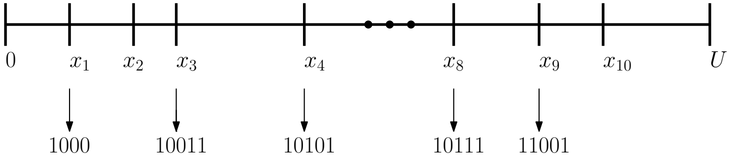

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Using the weak prefix data structure

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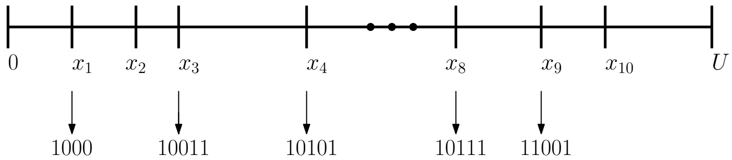

Answer = [4, 8]

- Given $[a, b]$, compute the longest common prefix of a and b in $O(1)$ time.
- $h(S) \cap [a, b]$ is non-empty iff:
  1. A largest point in $h(S)$ prefixed by $p \circ 0$ exists, and is not smaller than a, or
  2. A smallest point in $h(S)$ prefixed by $p \circ 1$ exists, and is not larger than b.

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- For simplicity, we only encode L-well separated point sets S.

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- How many L-well separated sets are there?
- Inductive construction: for the ith point, we have at least

$$U - 4L - 4(i-1)L = U - 4iL \text{ choices.}$$

Lemma

There are at least $M = (U - 4nL)^n / n!$ L-well separated sets of size n in a universe of size U. Encoding one such set requires $\lg M$ bits.

Size of encoding$(s, \epsilon) \geq \lg M$ gives the lower bound.

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