DISPERSING HASH FUNCTIONS

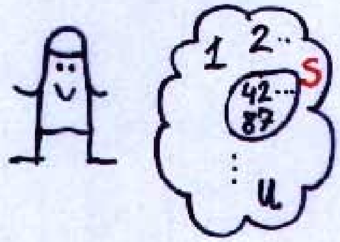
RASMUS PAGH
BRICS
UNIVERSITY OF AARHUS

PLAN:

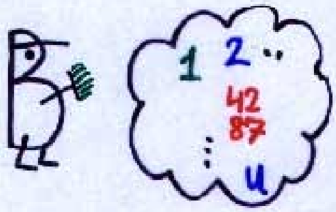
- "DISPERSING HASH FUNCTIONS" BY WAY OF A GAME.
- ELEMENT DISTINCTNESS: UNIVERSAL VS. DISPERSING.
- EXPLICIT CONSTRUCTIONS.
- "ALMOST PERFECT" HASH FUNCTIONS.

A TWO-PLAYER GAME

THE GAME "COLOURFUL" HAS ONE ROUND:



ALICE CHOOSES
$S \subseteq U = \{1, .., u\}$, $|S| = n$



INDEPENDENTLY, BOB
COLOURS U USING
COLOURS $\{1, \ldots, r\}$.

> **BOB'S AIM:** MAXIMIZE #COLOURS IN S
> (WORST CASE OVER ALL S)

A RANDOMIZED STRATEGY

- PICK EACH COLOUR AT RANDOM.
  ($u \cdot \log r$ BITS).

  ↑ BOB CAN'T AFFORD THIS :(

DISPERSING HASH FUNCTIONS

ASSUME $u \geq r^{1+\varepsilon}$, $r \geq n$. FOR $S \subseteq U$, $|S| = n$, A RANDOM FUNCTION $h: U \to \{1, \ldots, r\}$ HAS

$$E[|h(S)|] = n - \lambda, \quad \text{WHERE } \lambda = \Theta(n^2/r)$$

DEFINITION

A FAMILY $\{h_i\}$, $h_i: U \to \{1, \ldots, r\}$, IS C-DISPERSING IF $E_i[|h_i(S)|] \geq n - c\lambda$, FOR ALL $S \subseteq U$ WITH $|S| = n$

PSEUDO-RANDOMNESS

> IT IS USEFUL TO BE ABLE TO PICK AND STORE FUNCTIONS THAT "LOOK" LIKE RANDOM FUNCTIONS

MAIN CONCERNS:

- SIZE OF FAMILY
- EXPLICITNESS

PROPERTIES:

- LOW COLLISION PROBABILITY (UNIVERSALITY)
- k-WISE INDEPENDENCE
- NEAR-UNIFORMITY ON SUFFICIENTLY LARGE SETS (EXTRACTORS).

MOTIVATING EXAMPLE

ELEMENT DISTINCTNESS:
ARE $x_1, \ldots, x_n \in \{1, \ldots, 2^b\}$ DISTINCT?

ALGORITHM:

1. PICK $h : \{1, \ldots, 2^b\} \to \{1, \ldots, n^2\}$ AT RANDOM FROM A $(n/\log n)$-DISPERSING FAMILY.
2. PUT $x_1, \ldots, x_n$ IN BUCKETS ACCORDING TO $h(x_1), \ldots, h(x_n)$. (RADIX SORT)
3. FOR EACH BUCKET, PUT THE ELEMENTS IN A BINARY SEARCH TREE (STOP IF DUPLICATE)

ANALYSIS:

- $O(n/\log n)$ DISTINCT ELEMENTS IN NON-TRIVIAL BUCKETS, $\Rightarrow$ $O(n)$ TIME
- LINEAR SPACE

DISPERSING VS. (ALMOST) UNIVERSAL FAMILIES

	SIZE	
UNIVERSAL:	$O(r \cdot \log u)$	$\Omega(r \cdot \log_r u)$
C-DISPERSING:	$O(r \cdot \log_c(u)/cn)$ NON-CONSTRUCTIVE	$\Omega(r \cdot \log_{2r/n}(u)/cn)$

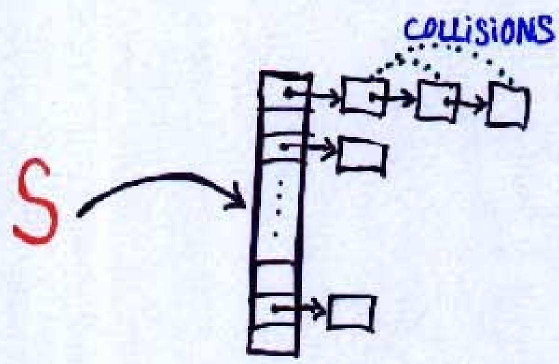
EXAMPLES, CONTINUED

- ELEMENT DISTINCTNESS ALGORITHM NEEDS
  $$O(\log(n^2 \cdot \log_{n/\log n}(2^b)/(n^2/\log n))) = O(\log b)$$
  RANDOM BITS (RATHER THAN $O(\log n + \log b)$).
- BOB NEEDS, FOR $c = 1+\varepsilon$, $O(\log\log u)$ RANDOM BITS.

EXPLICIT CONSTRUCTION I

OBSERVATION (FREDMAN ET AL):

$|h(S)| \geq n - \#$ COLLIDING PAIRS

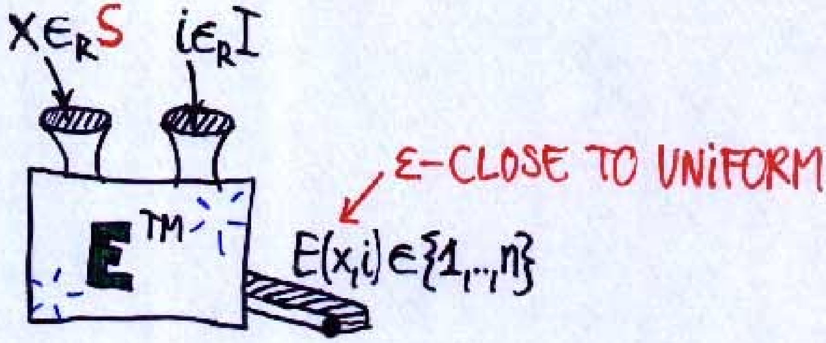


$\boxed{\text{(ALMOST) UNIVERSAL FAMILIES ARE } O(1)\text{-DISPERSING}}$

EXPLICIT CONSTRUCTION II_A

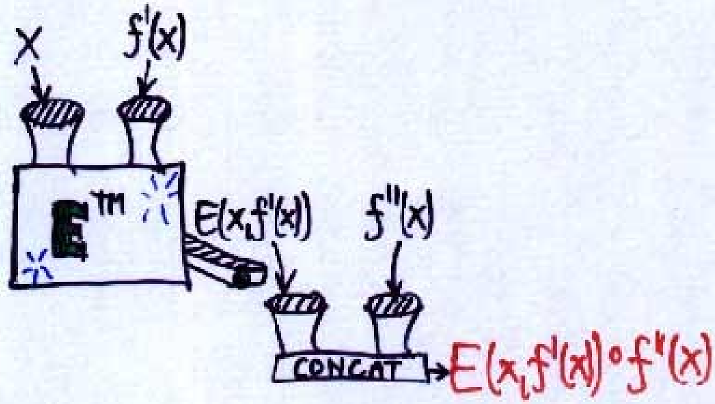
EXTRACTOR (SIMPLIFIED):

FOR ANY $S \subseteq U$ WITH $|S| = n$



EXPLICIT CONSTRUCTION II_B

- SELECT f' FROM A STRONGLY UNIVERSAL FAMILY, $f' : U \to I$.
- SELECT $f'' : U \to \{1, \dots, r/n\}$ FROM A UNIVERSAL FAMILY.



ANALYSIS:

LOOK AT COLLISIONS OUTSIDE A SET OF "BAD" POINTS, TO WHICH FEW ELEMENTS MAP.

A DIFFERENT VIEW OF EXTRACTORS

EXTRACTOR WITH OPTIMAL $\log |I|$

SEED LENGTH

NON-TRIVIAL

$O(1)$-DISPERSING FAMILY WITH OPTIMAL SAMPLE COMPLEXITY

WHEN DESIGNING EXPLICIT EXTRACTORS, ONE MAY W.L.O.G. CONSIDER COLOURFUL STRATEGIES

DETOUR: "ALMOST PERFECT" HASHING

- MINIMAL PERFECT HASH FUNCTION:

  $h: U \to \{1, \ldots, n\}$

  $|h(S)| = |S| = n$

  PROGRAM SIZE: $\Omega(n)$ BITS

- "ALMOST PERFECT" HASH FUNCTION:

  $h: U \to \{1, \ldots, n\}$

  $|h(S)| \geq \frac{2}{3}|S| = \frac{2}{3}n$

  PROGRAM SIZE: $\Omega(n)$ BITS

CONCLUSION

DISPERSING HASH FUNCTIONS

+ SEVERAL (MANY?) APPLICATIONS
+ POTENTIALLY LOW SAMPLE COMPLEXITY

¿ EXPLICIT CONSTRUCTION PROBABLY DIFFICULT (AT LEAST FOR $c = O(1)$).

¿ EVER AS FAST AS UNIVERSAL HASH FUNCTIONS?

THAT'S ALL