From Independence to Expansion and Back Again

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Scalable Similarity Search

Topic of this talk:

- Upper bounds on the space-time tradeoff of k-independent functions in the word RAM model

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$$\Pr_{f \in \mathcal{F}}[f(x_1) = y_1, f(x_2) = y_2, \ldots, f(x_k) = y_k] = r^{-k}$$

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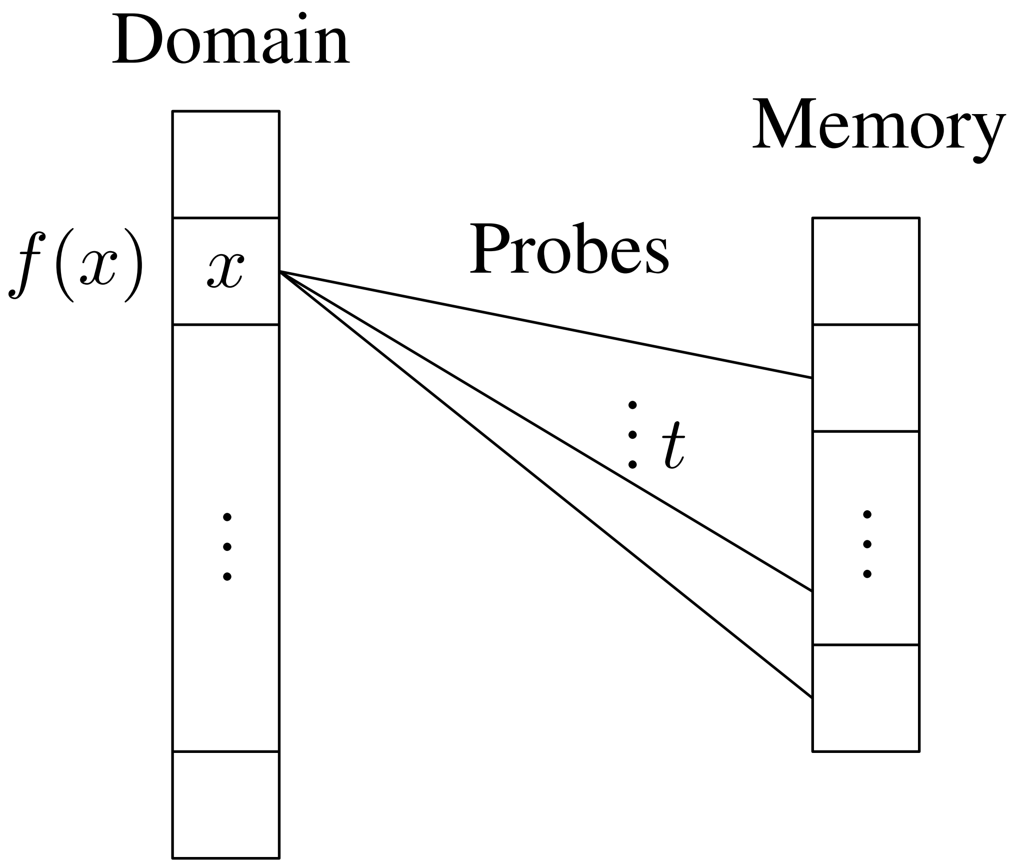
- Tradeoff:
  - Space used to represent $f \in \mathcal{F}$
  - Time used to evaluate $f \in \mathcal{F}$

Lower bound

Theorem [Siegel'89] A data structure for representing a k-independent function $f : [u] \to [r]$ with evaluation time $t < k$ must use at least $ku^{1/t}$ words of space



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Main result (vanilla version)

Randomized data structure for representing a k-independent function $f : [u] \to [r]$ with a space usage of $O(ku^{1/t}t)$ and evaluation time $O(t \log t)$

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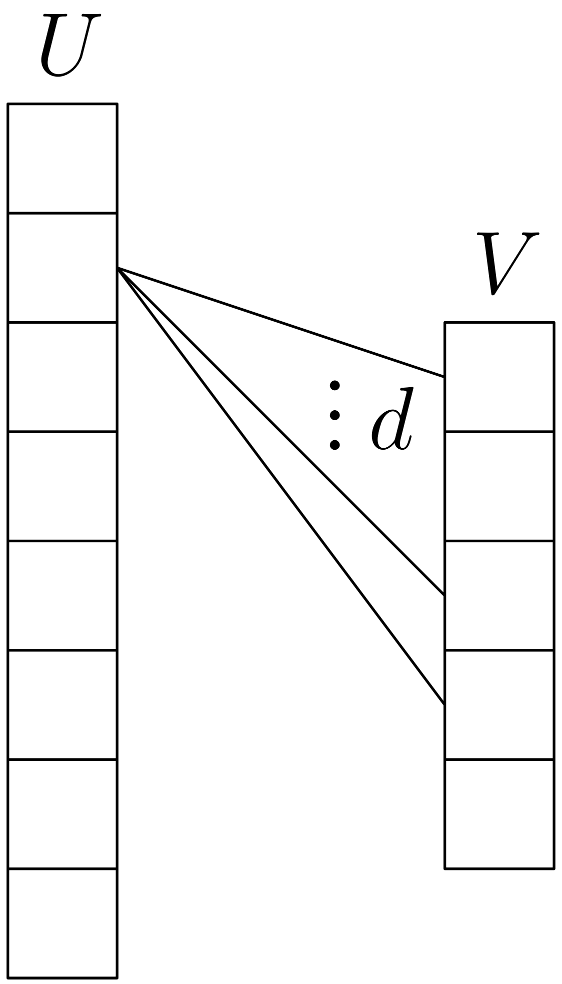
Previous results

Reference	**Space**	**Time**
Polynomials [Joffe'74]	$O(k)$	$O(k)$
Graph powering [Siegel'89]	$O(k^t u^{1/t})$	$O(t)^t$
Recursive tabulation [Thorup'13]	$O(\text{poly}\, k + u^{1/t})$	$O(t^{\log t})$

- Constructions of k-independent families of functions based on bipartite expander graphs
- Neighbor function $\Gamma : U \to V^d$



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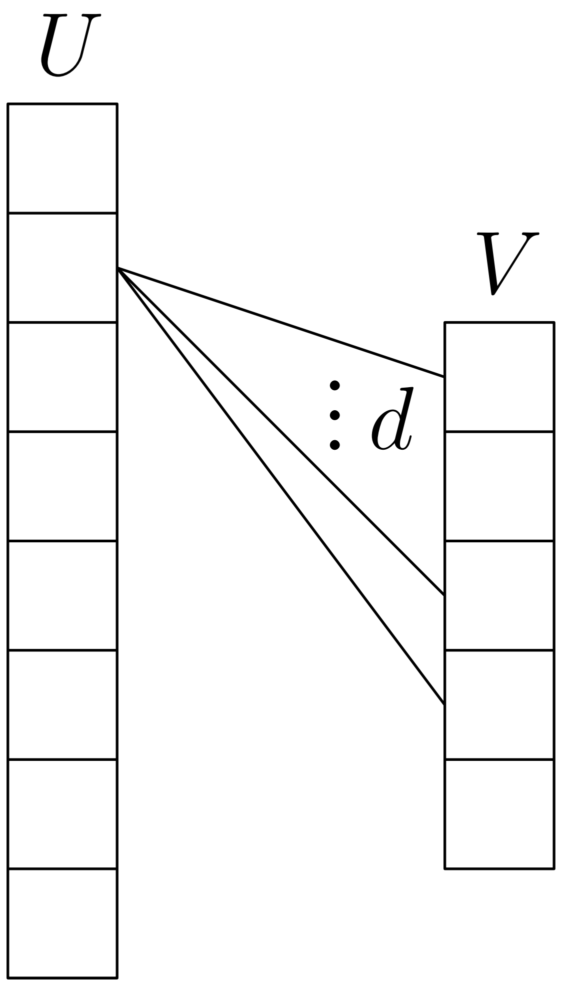
Definition A bipartite graph Γ is k-unique if for every $S \subseteq U$ with $|S| \leq k$ there exists $v \in V$ with exactly one neighbor in S



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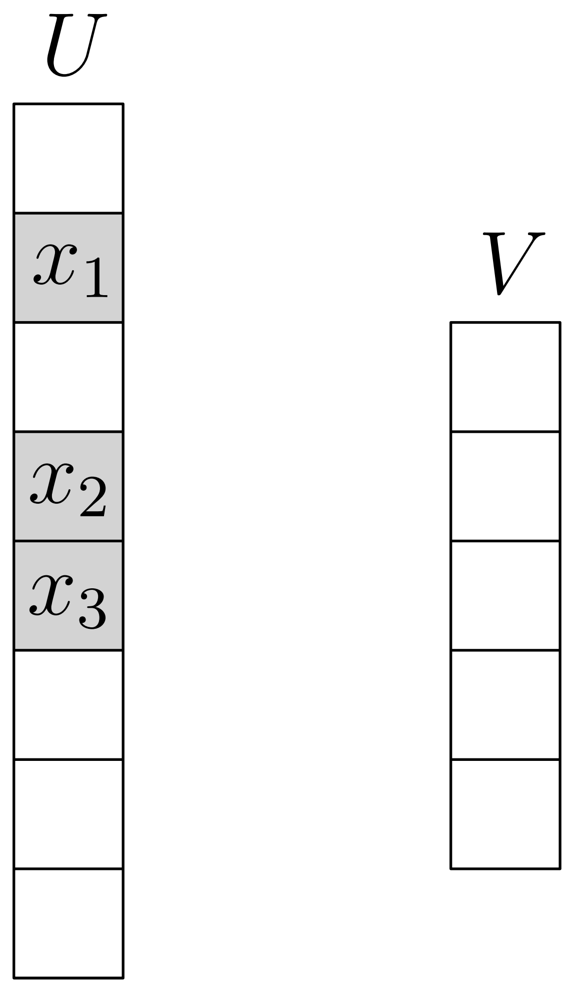

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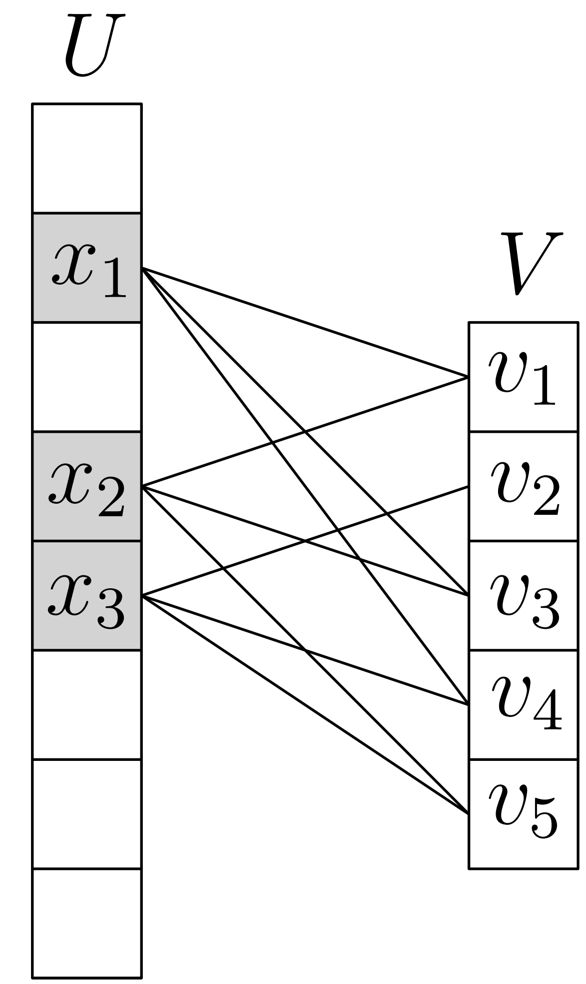

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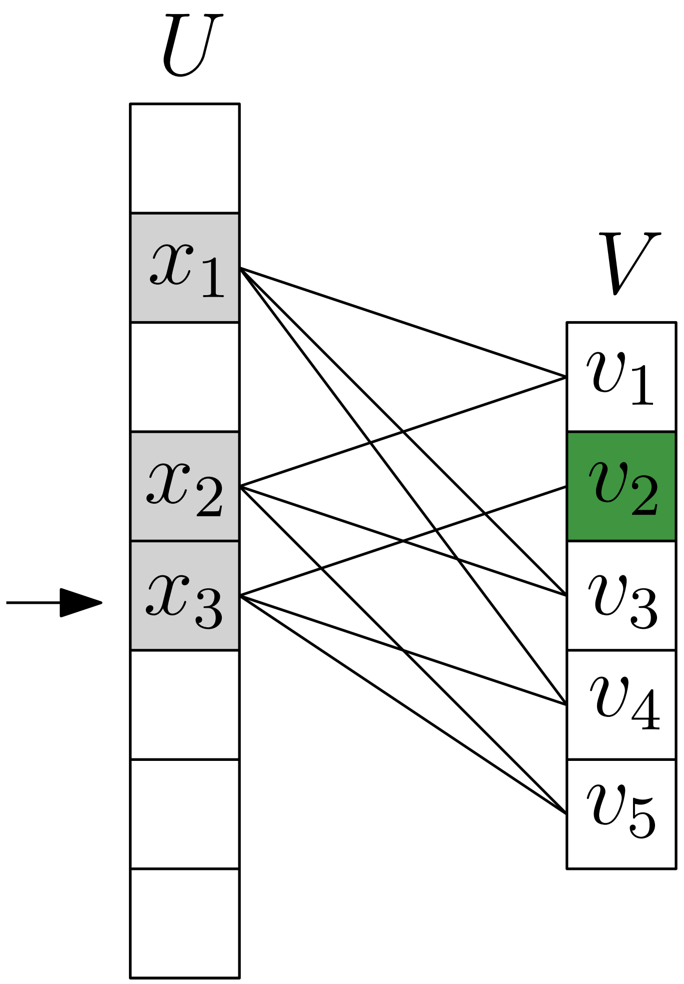

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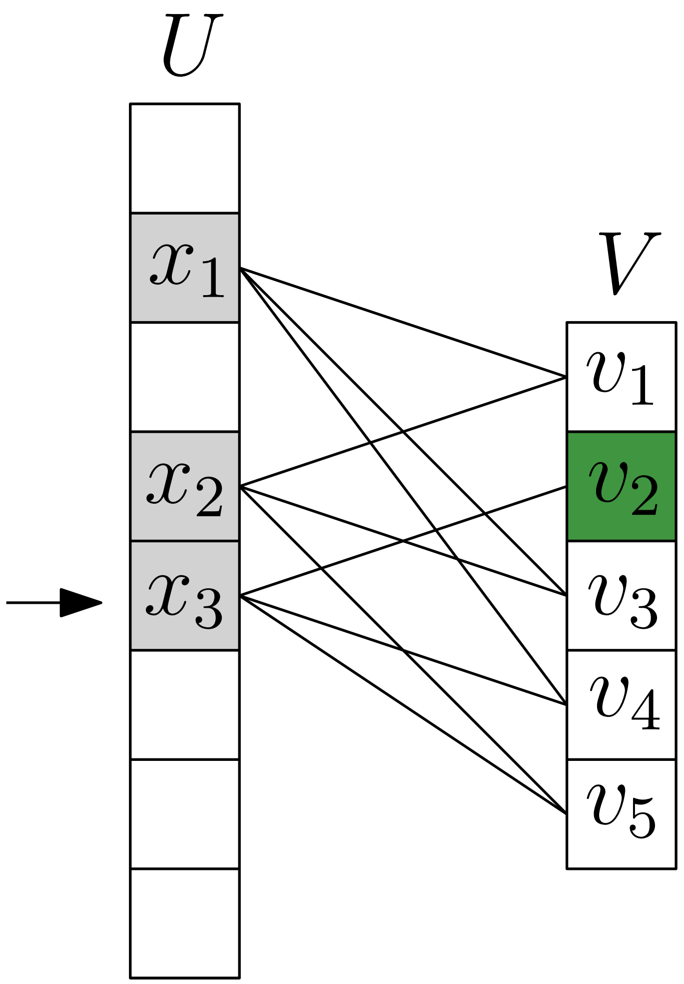

Lemma [Siegel'89] Let $\Gamma : U \to V^d$ be k-unique and $h : V \to [r]$ be a random function. Then $f(x) = \sum_i h(\Gamma(x)_i) \bmod r$ defines a k-independent family of functions

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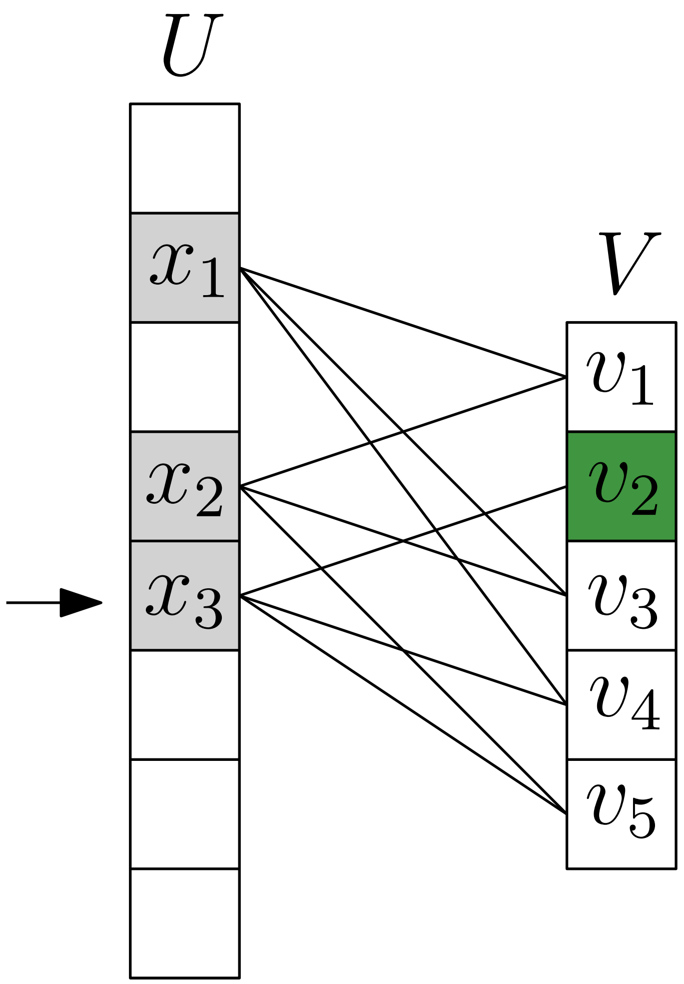

- Peeling argument example:

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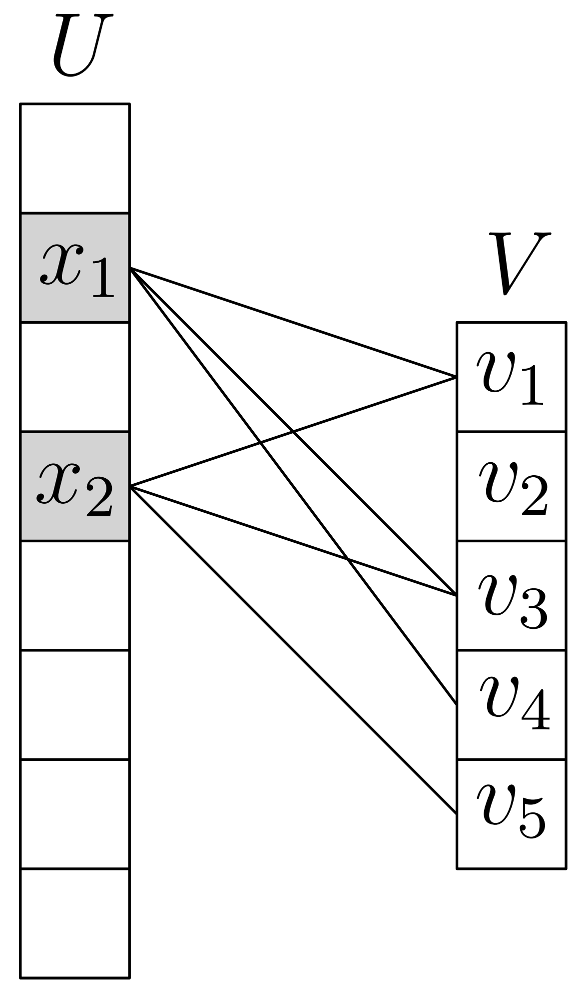

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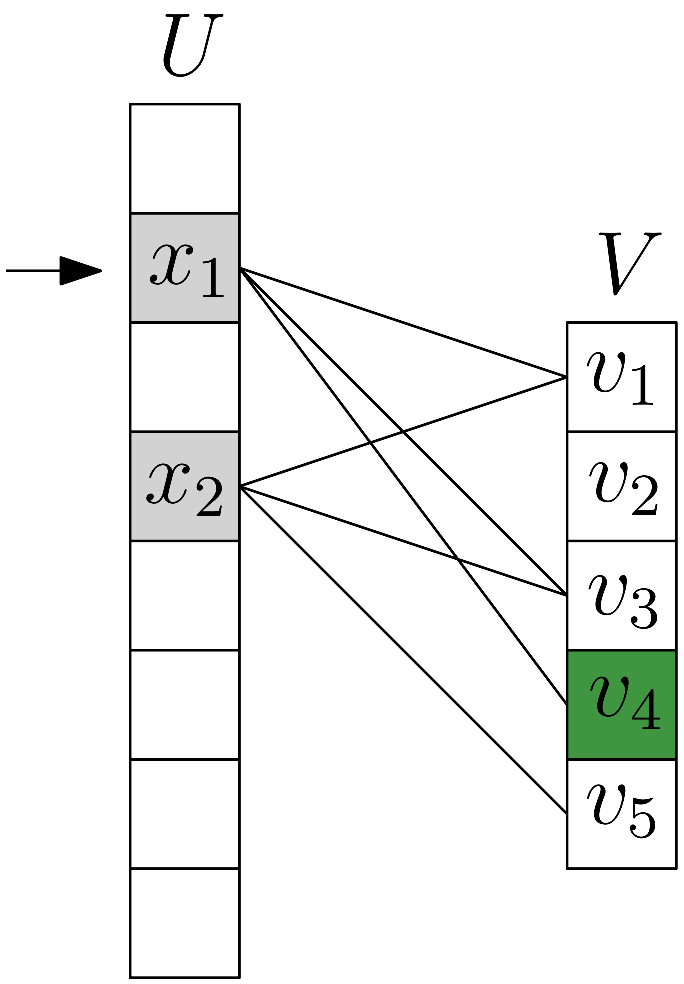

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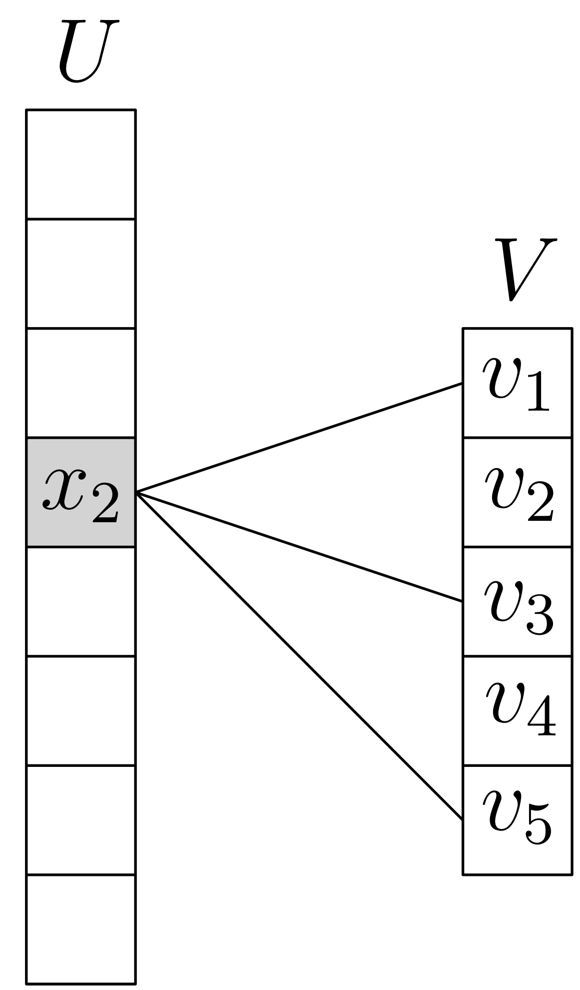

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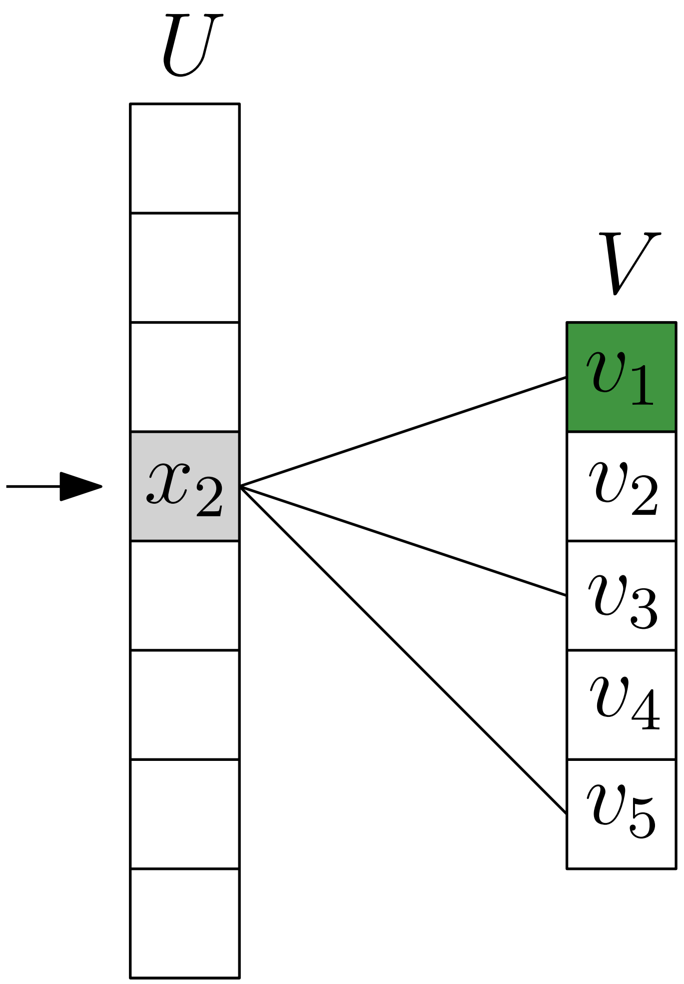

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From expansion to independence

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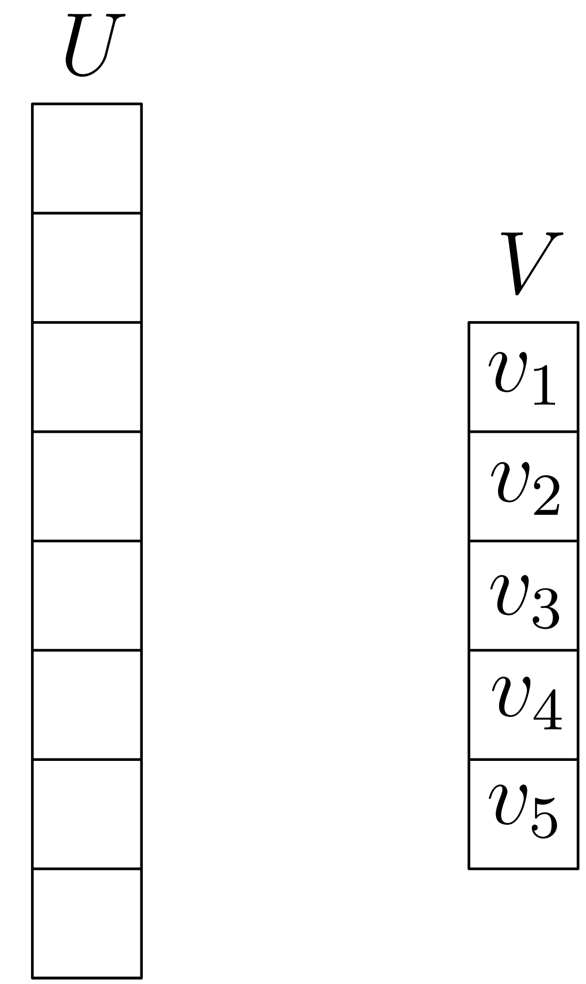

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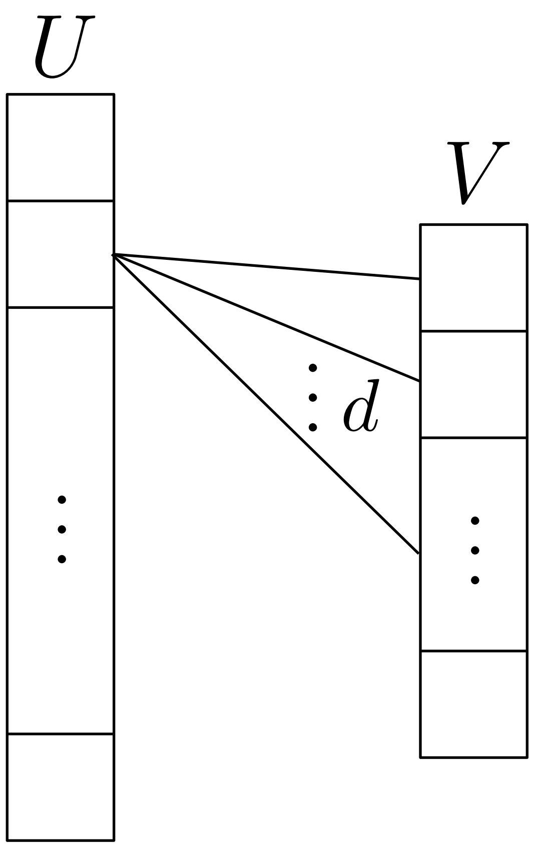

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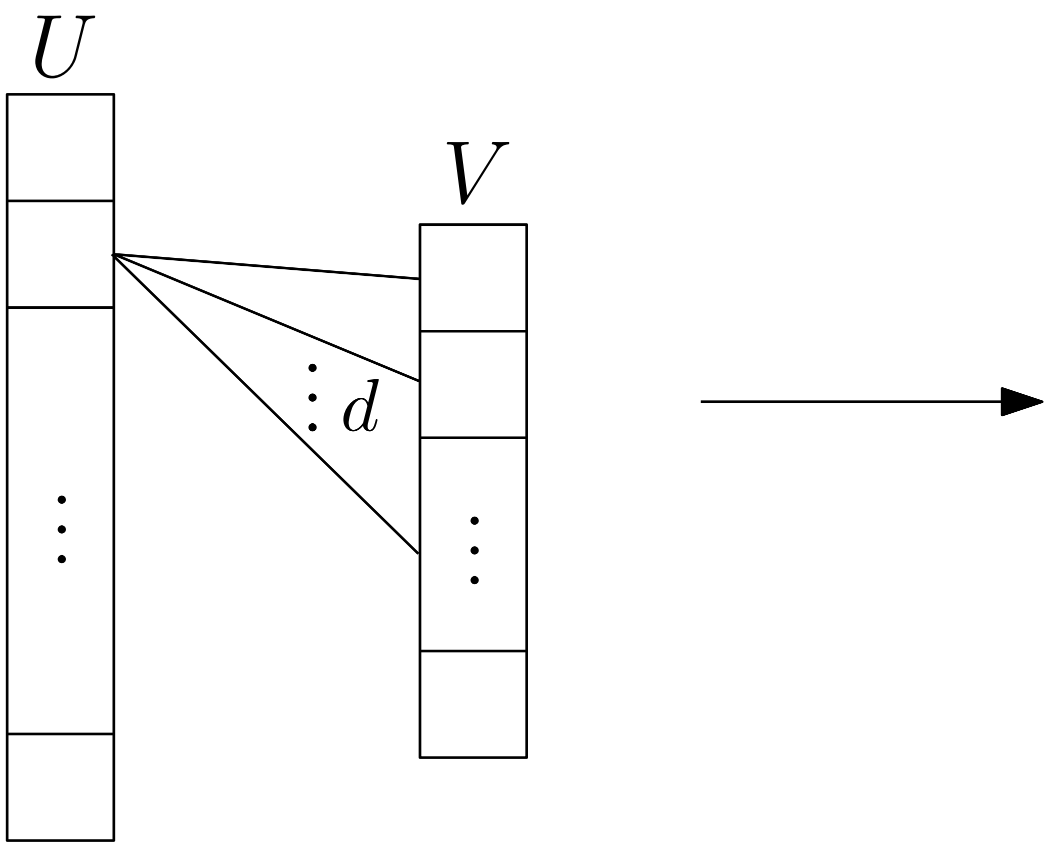

k-independent family $\mathcal{F}$

$f(x) = \sum_i h(\Gamma(x)_i) \bmod r$
$h : V \to [r]$ is random

Space $|V|$
Time d

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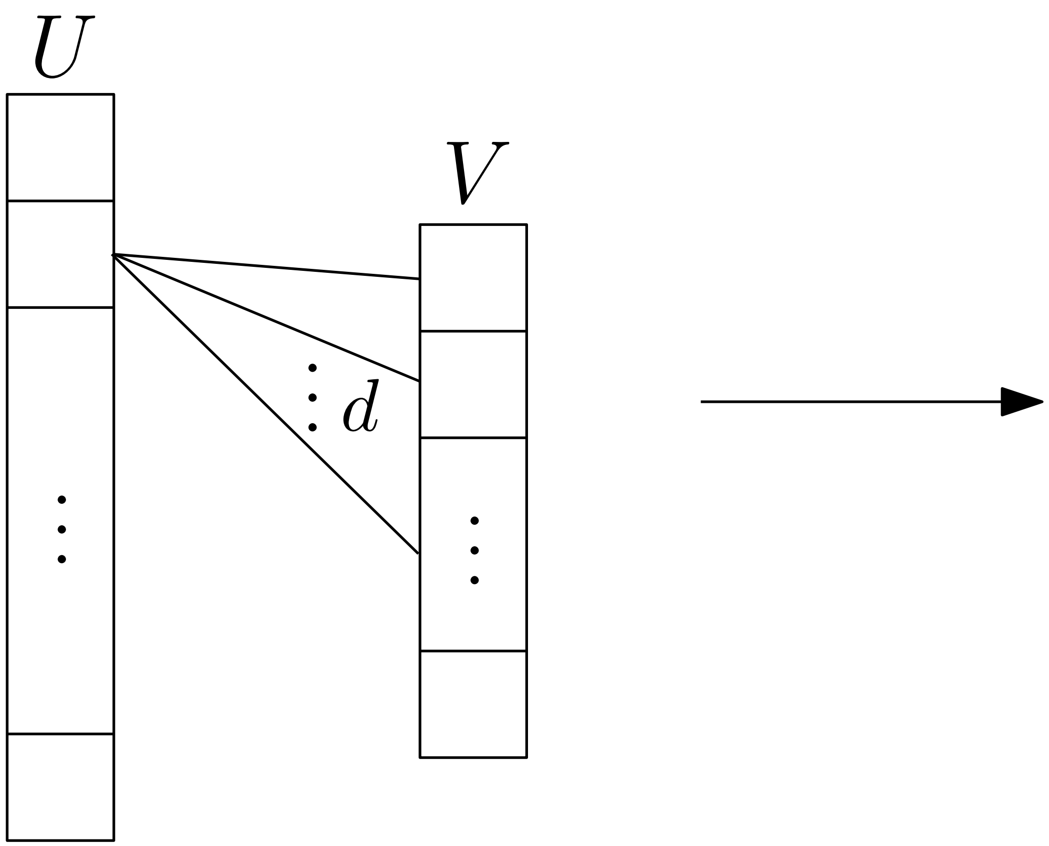

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Lemma [Siegel'89] Most graphs Γ give an optimal space-time tradeoff for k-independent hashing

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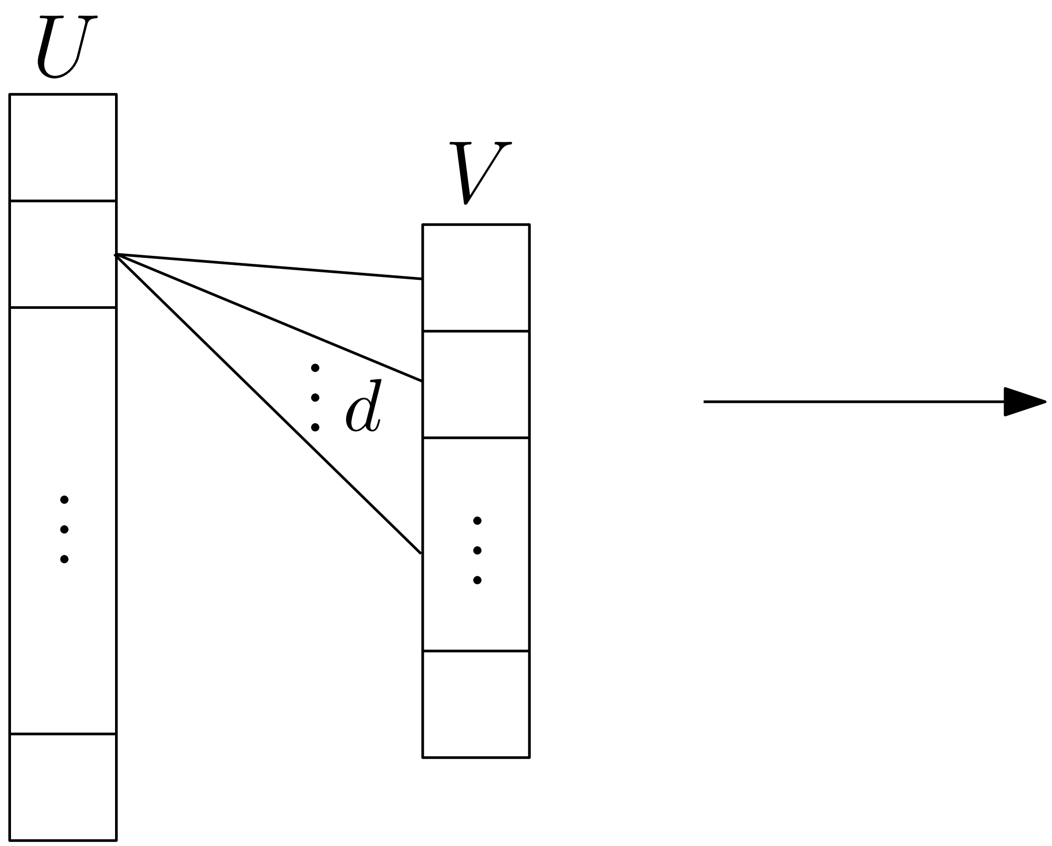

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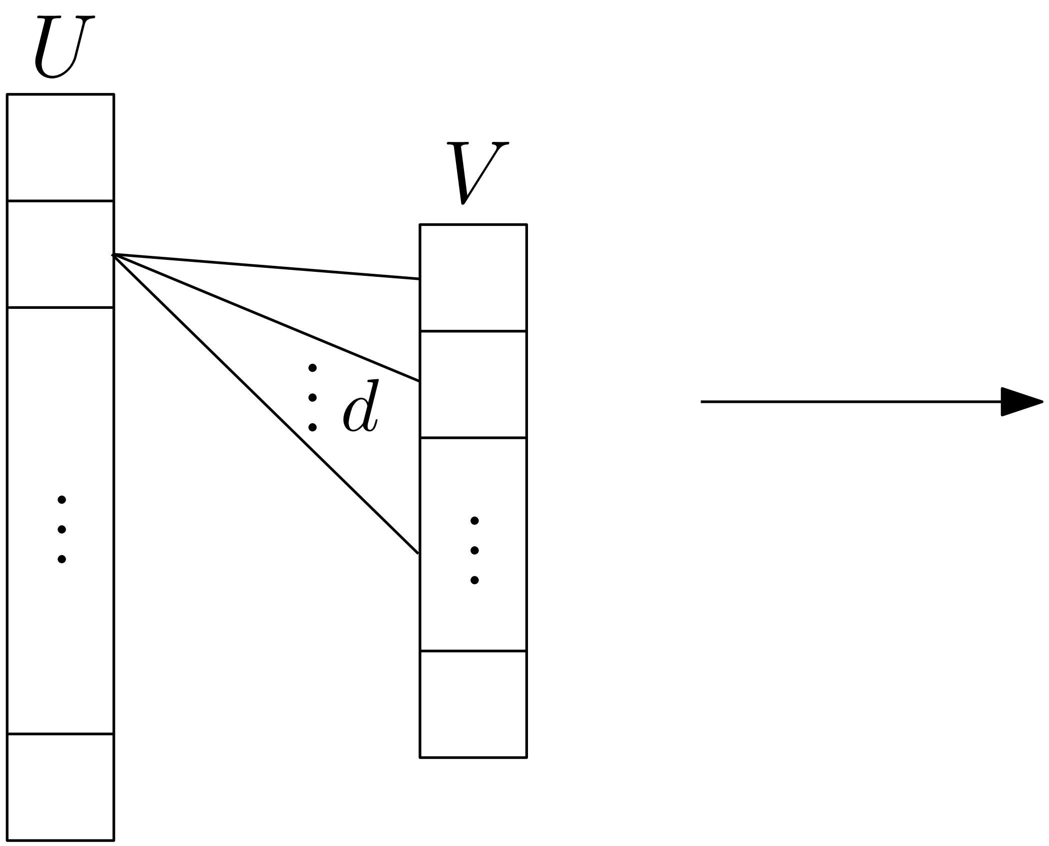

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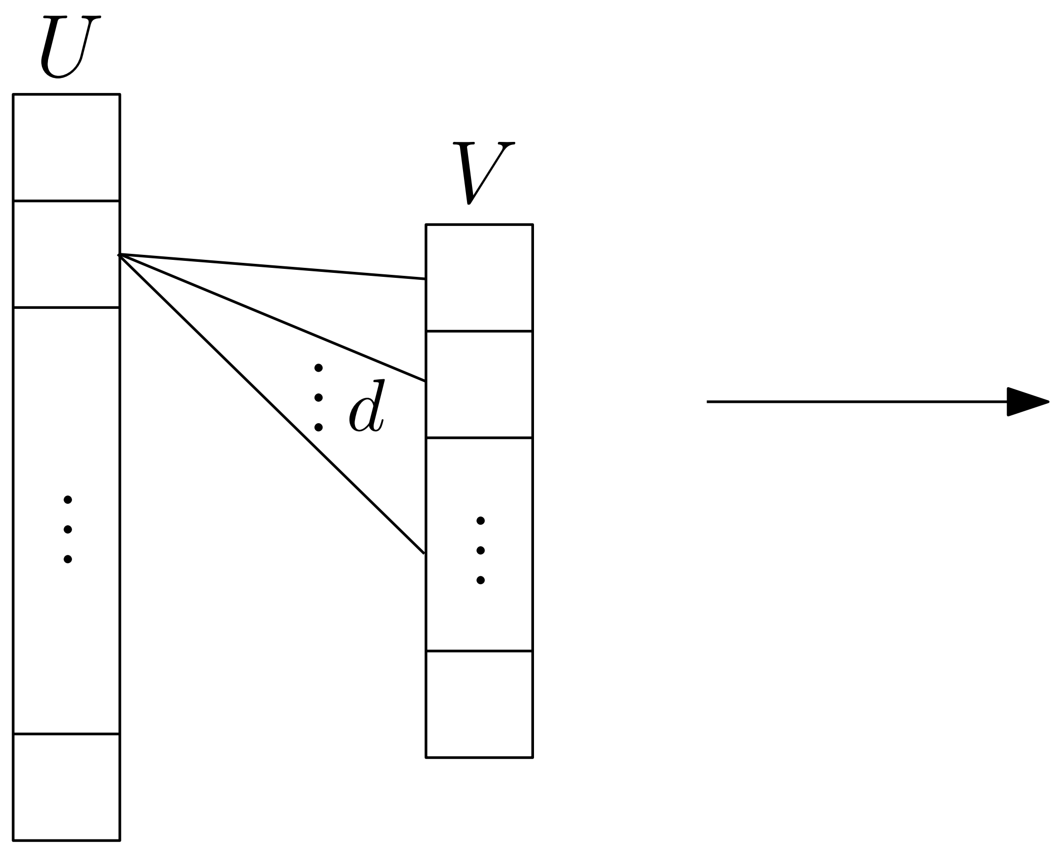

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Lemma [Siegel'89] Most graphs Γ give an optimal space-time tradeoff for k-independent hashing

- Explicit k-unique graphs with optimal parameters are not known
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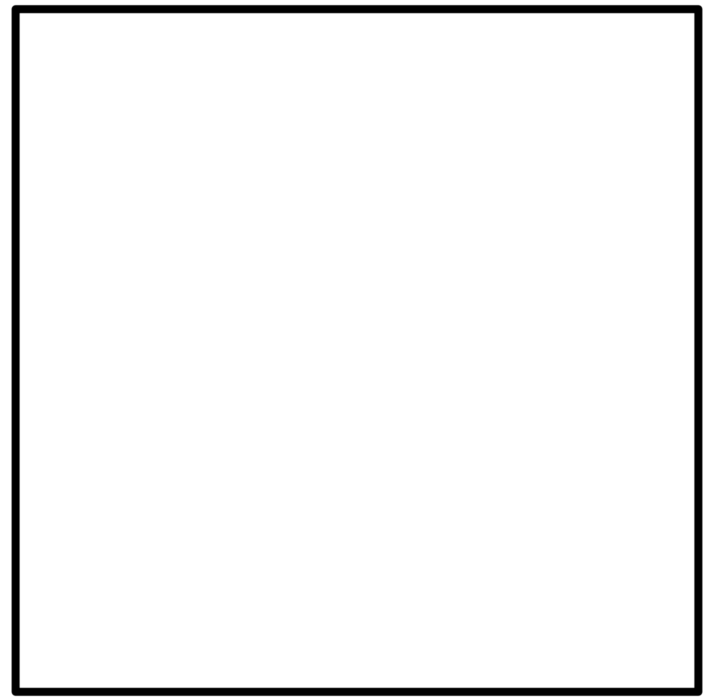
- Verifying that a given Γ is k-unique is infeasible

Set of all functions $\Gamma : U \to V^d$



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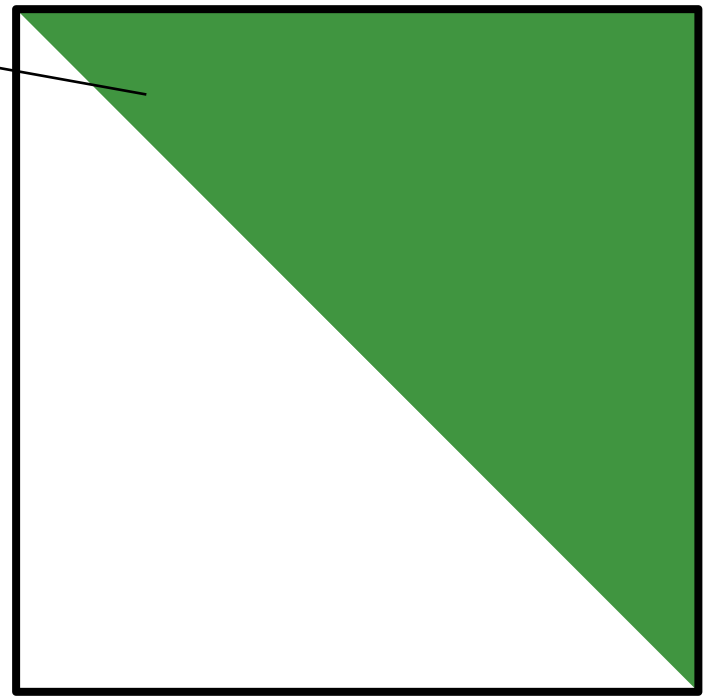
Optimally k-unique



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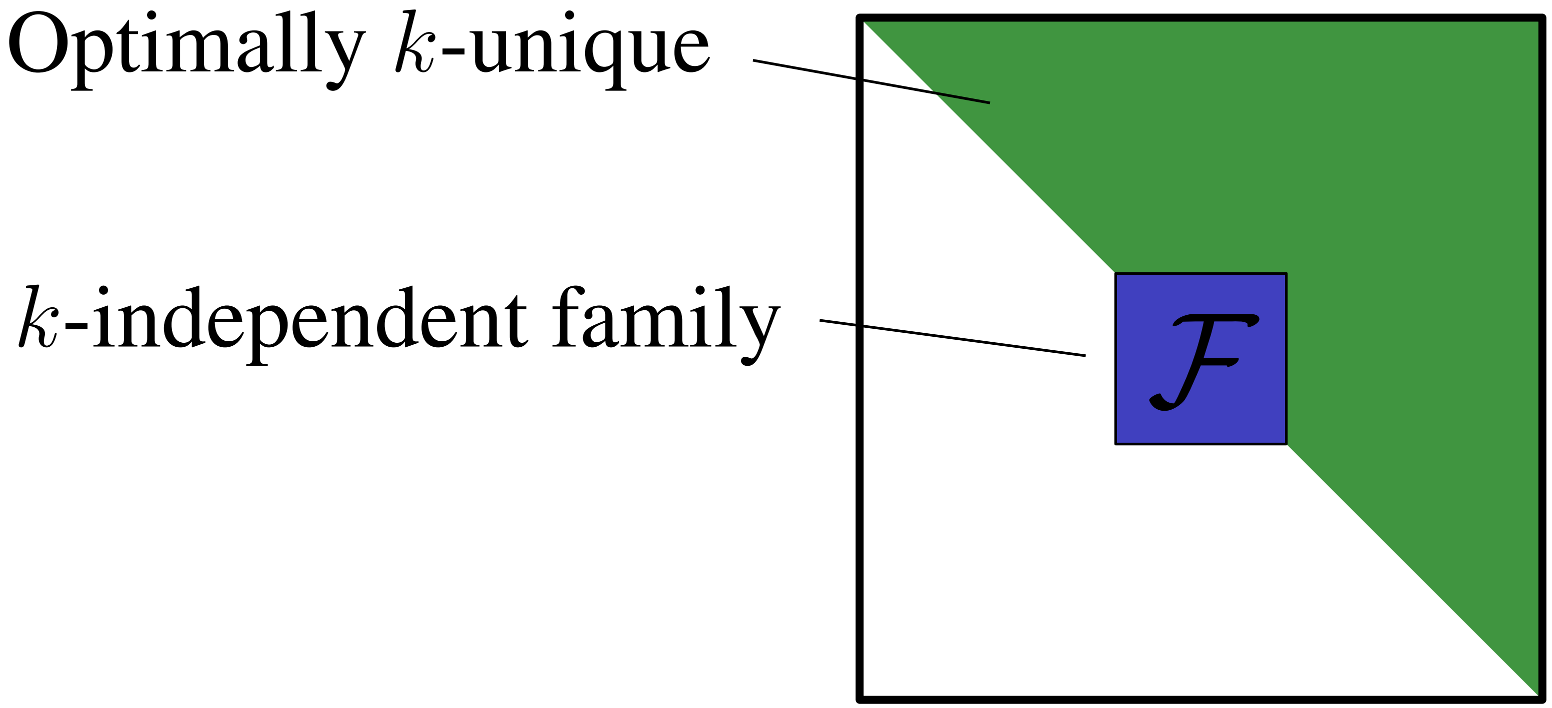

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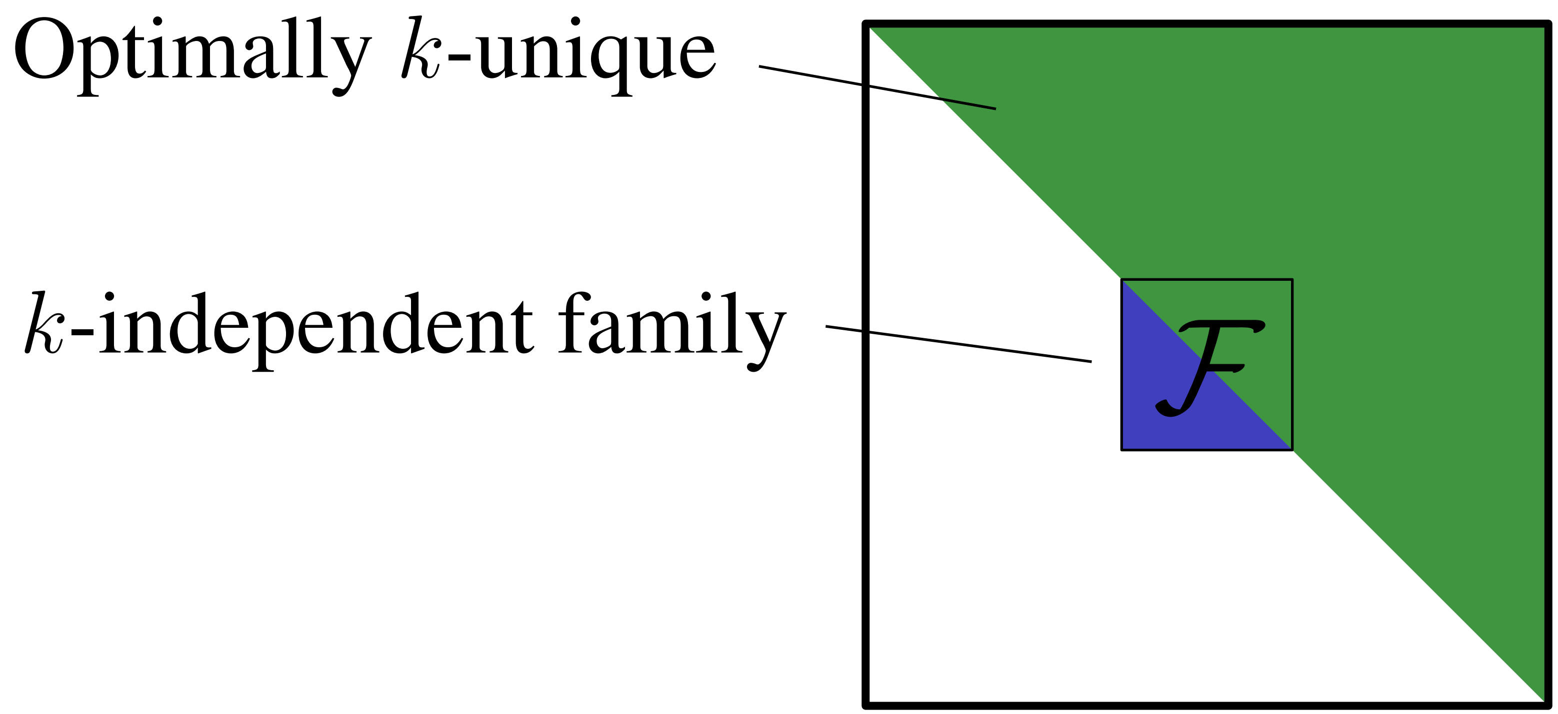

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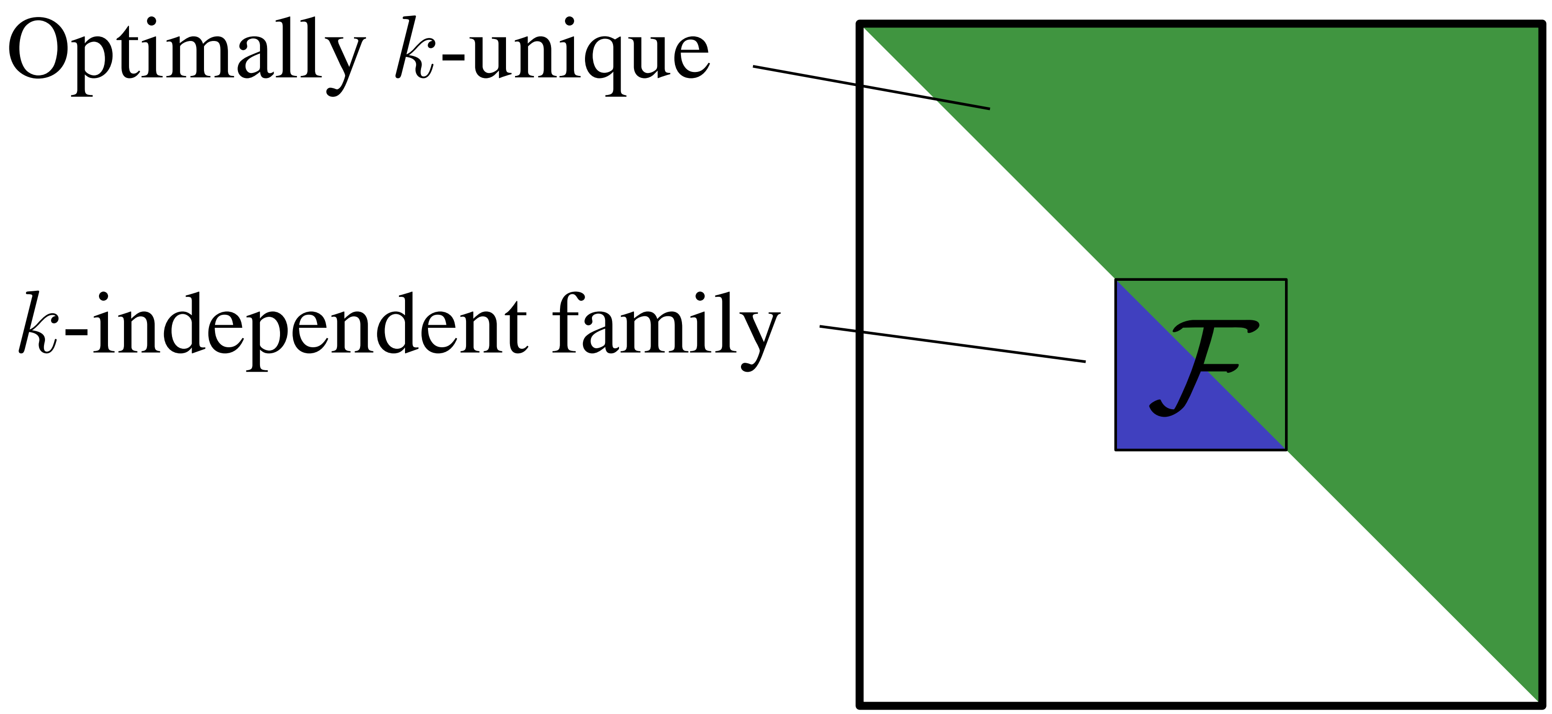

Lemma A k-independent function $\Gamma : U \to V^d$ with $|U| = u$, $|V| = O(ku^{1/t}t)$ and $d = O(t)$ is k-unique with high probability.

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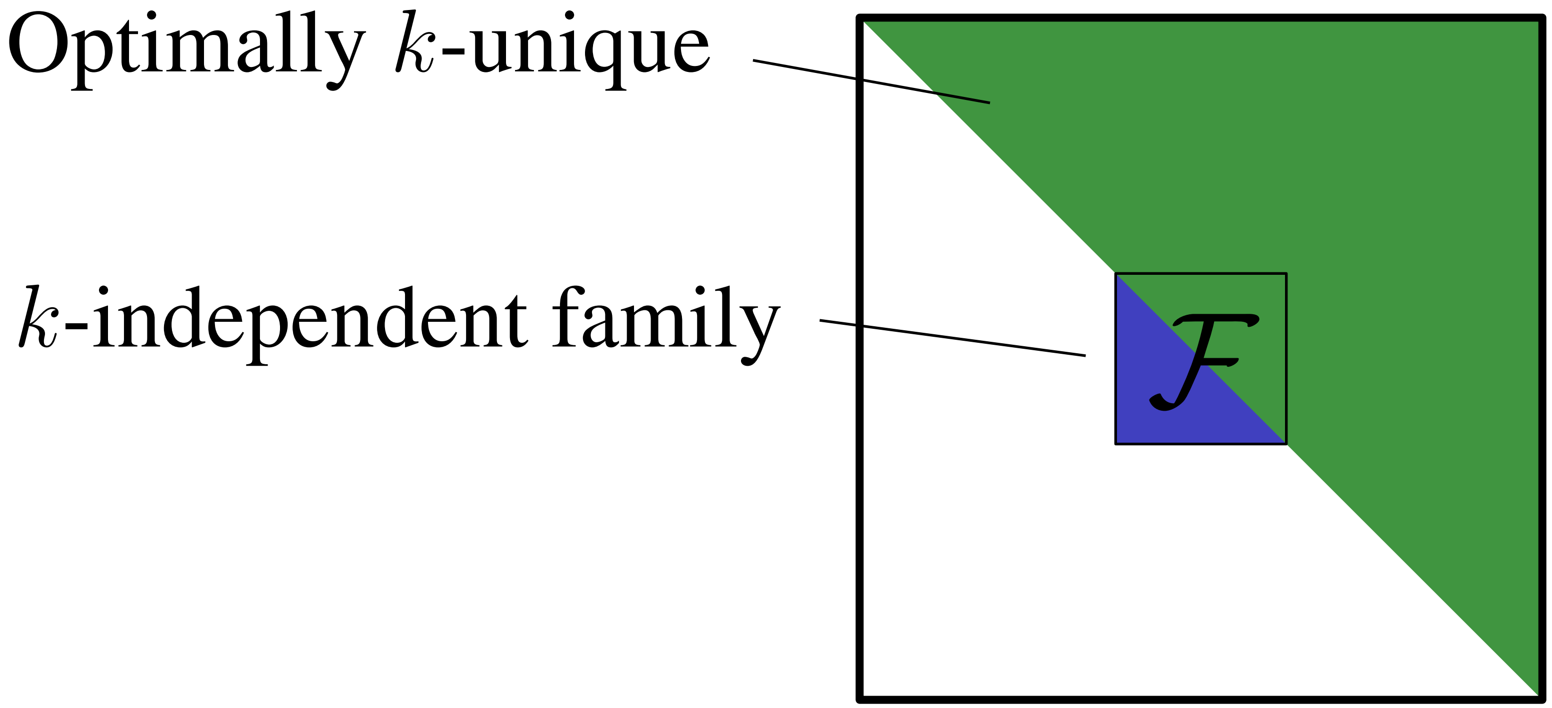

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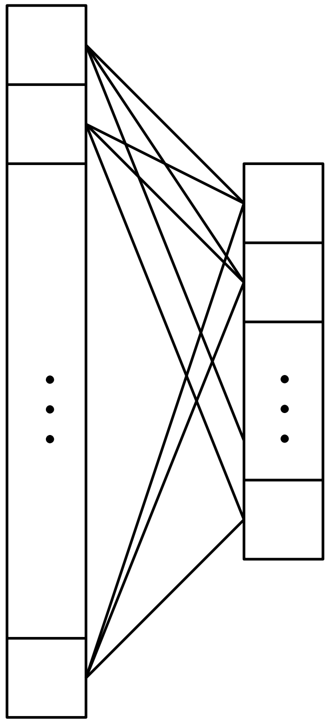
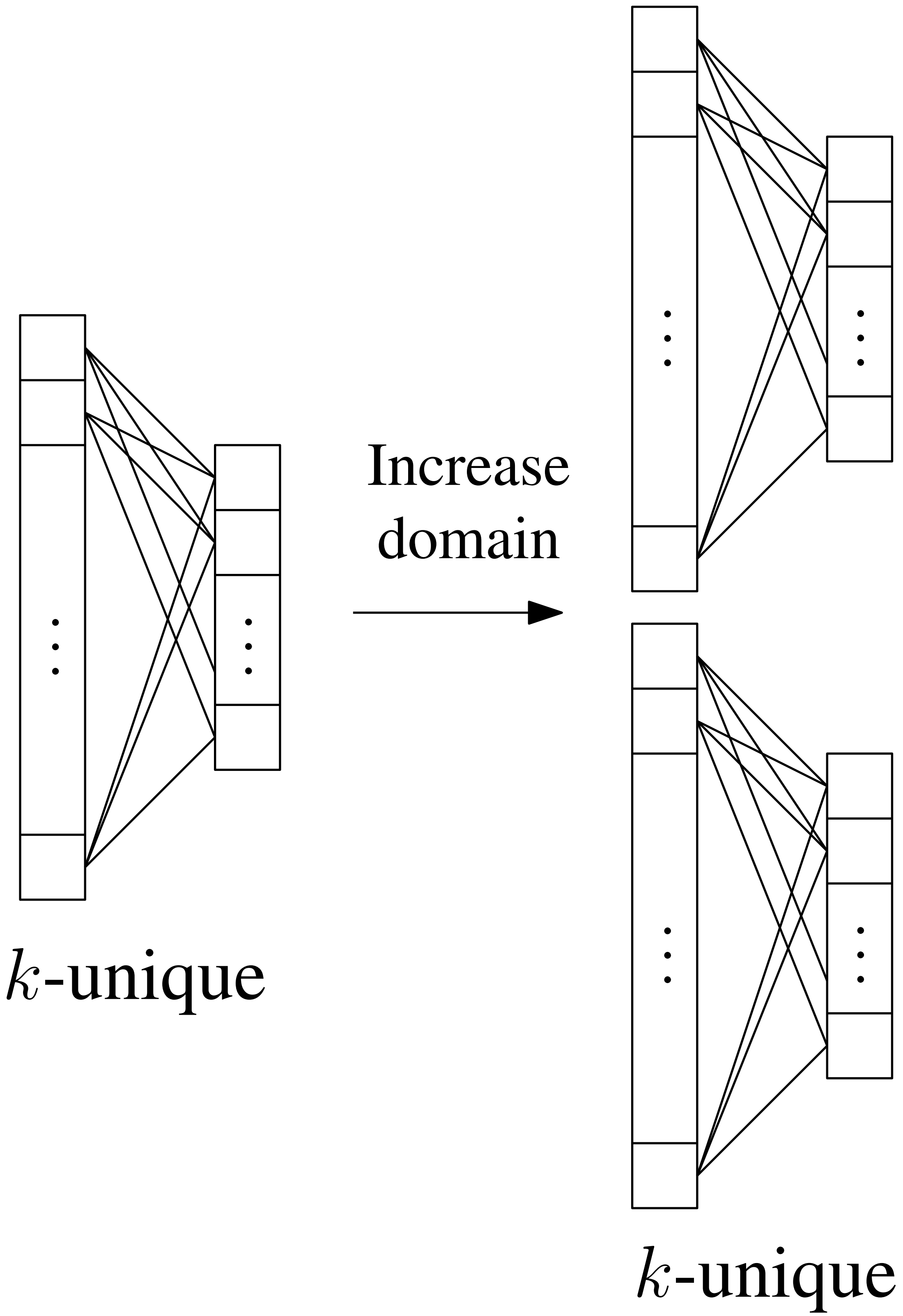
k-unique		k-independent family		k-unique whp.
Γ	$\longrightarrow$	$\mathcal{F}$	$\longrightarrow$	$\Gamma \in \mathcal{F}$



k-unique



k-unique
Increase domain
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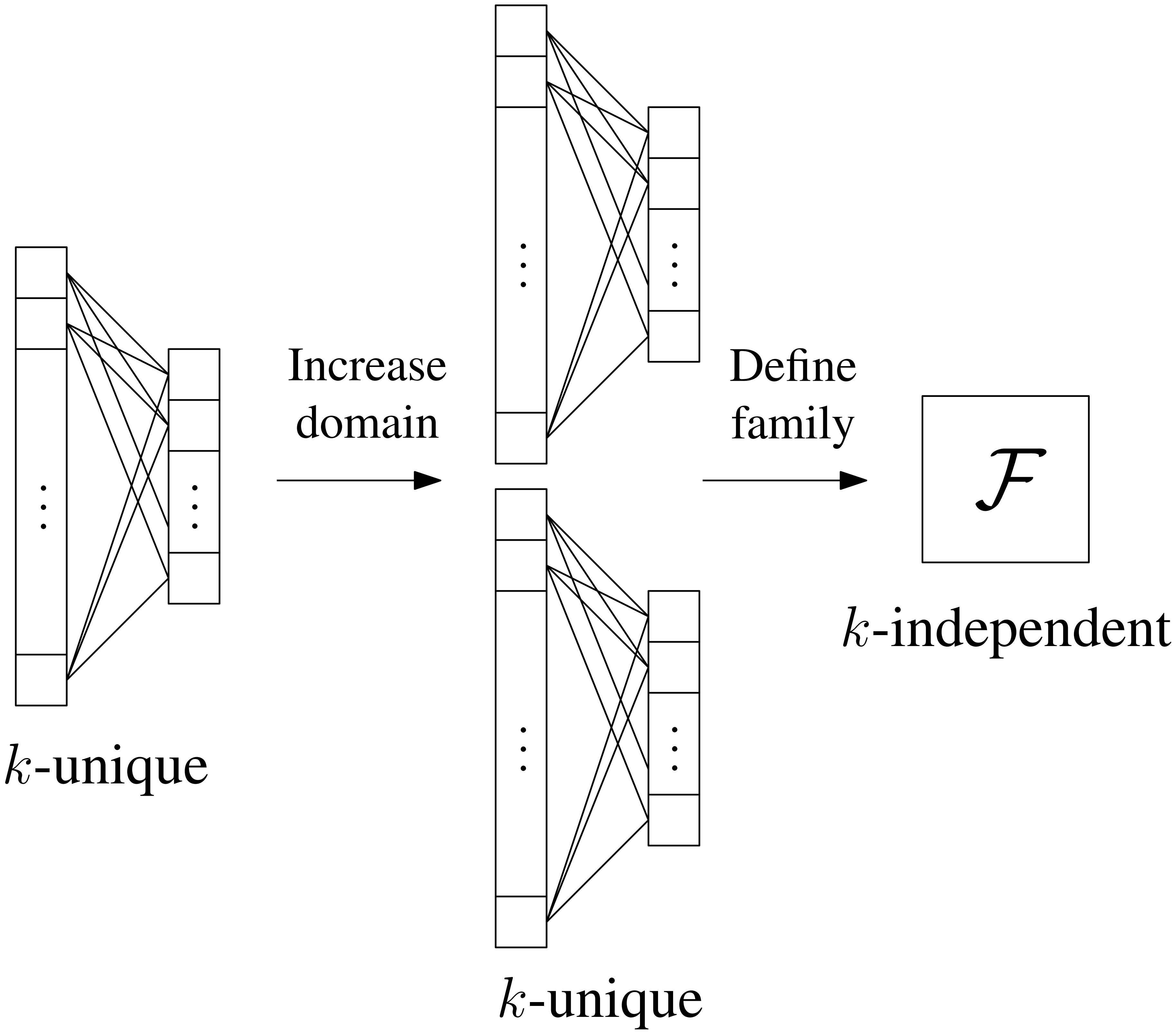
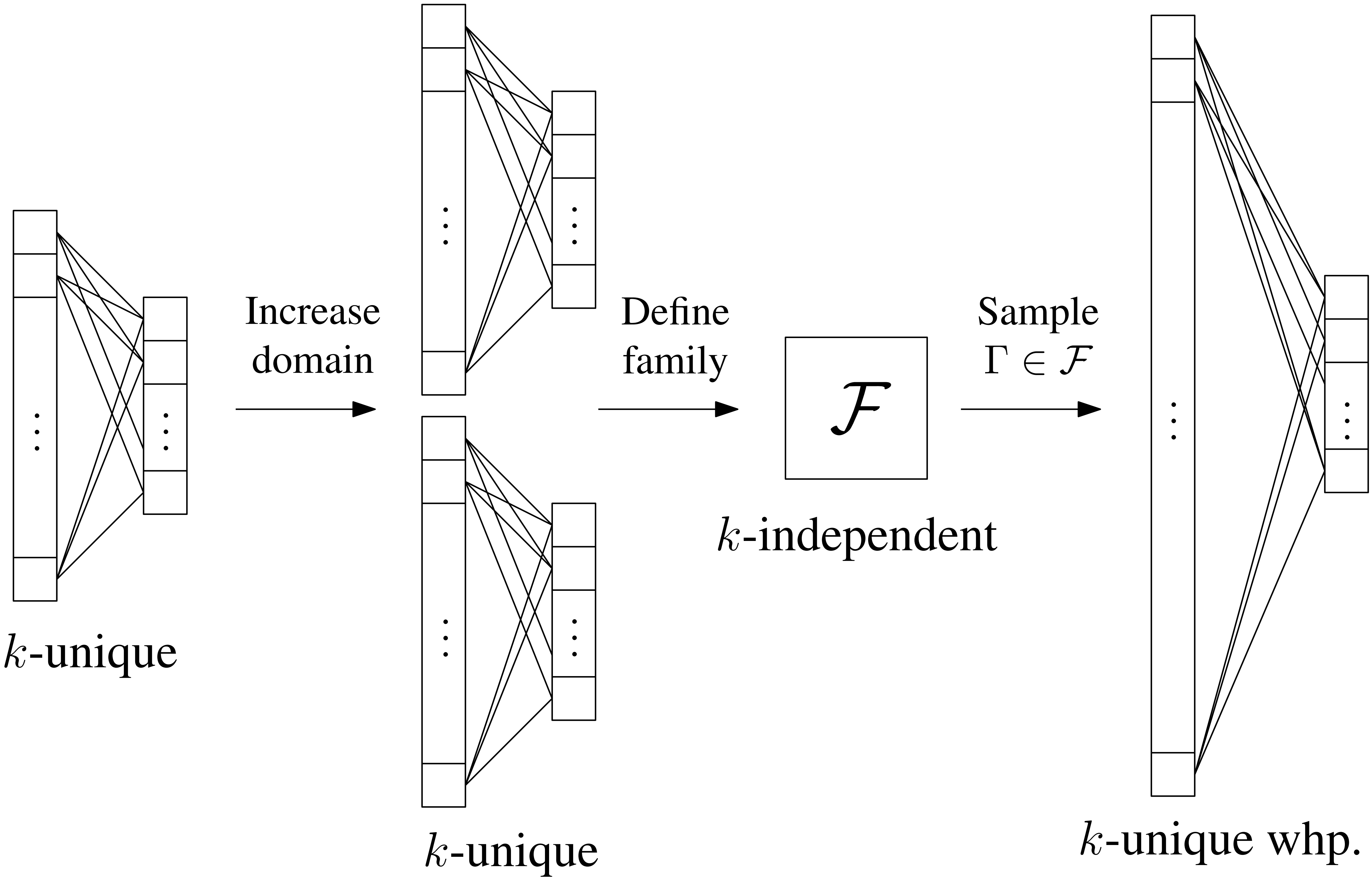

Increase
domain
Define
family
$\mathcal{F}$
k-unique
k-unique
k-independent



k-unique
Increase domain
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Define family
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k-independent
Sample $\Gamma \in \mathcal{F}$
k-unique whp.

- View $x \in [u]$ as a string of characters from an alphabet Σ

$$\Gamma(ax) = \bigoplus_i h(a, \Gamma(x)_i), \qquad a \in \Sigma, ax \in \Sigma^*$$

$h : \Sigma \times V \to V^d$ is random

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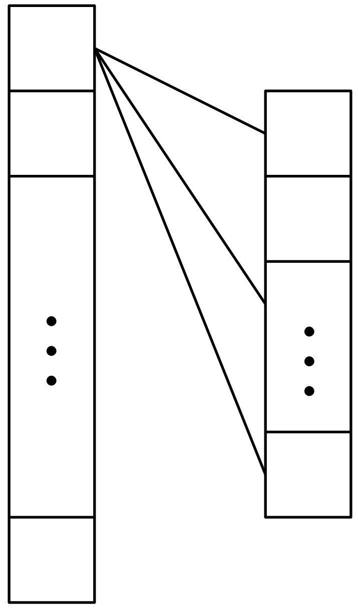

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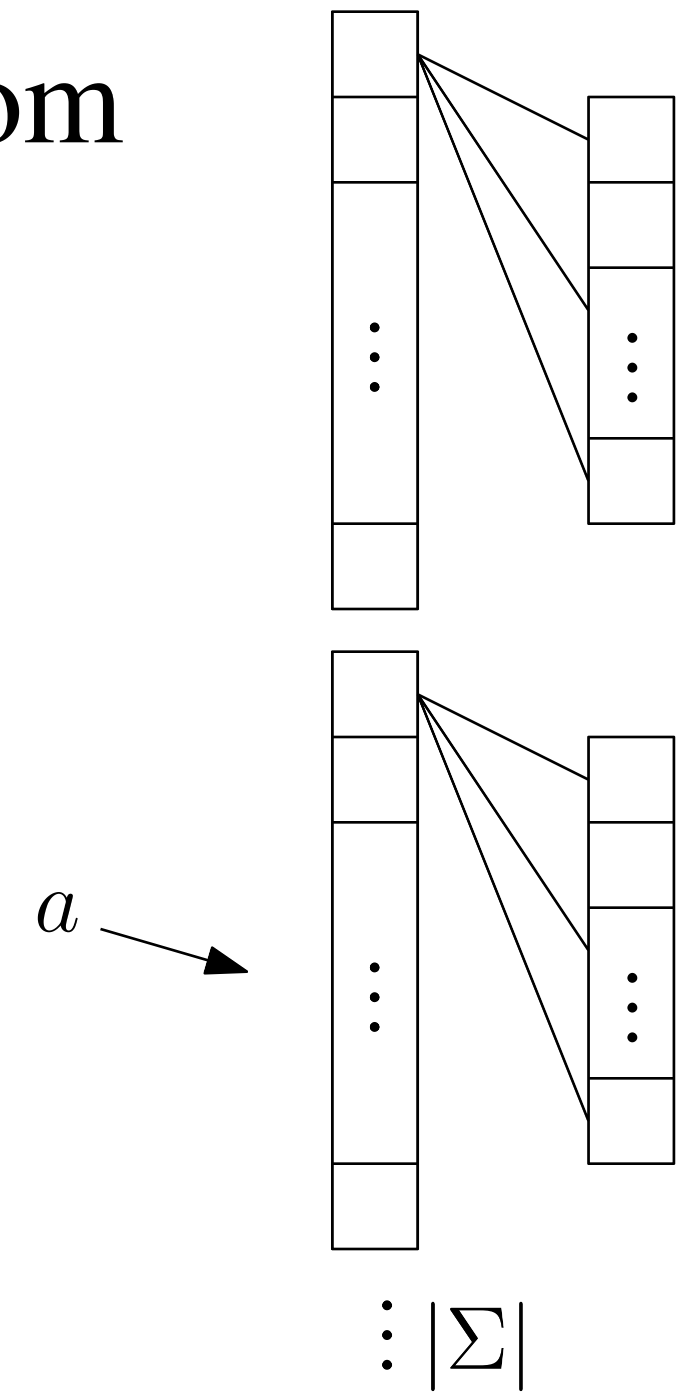

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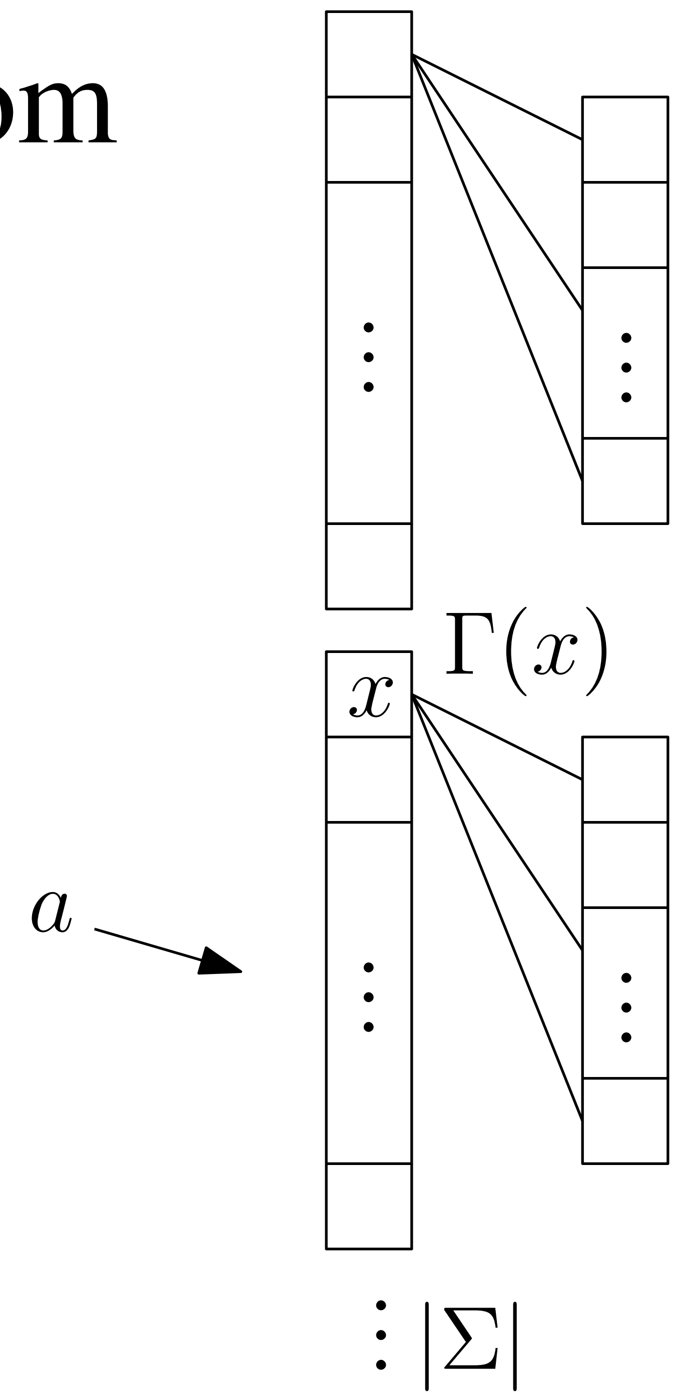

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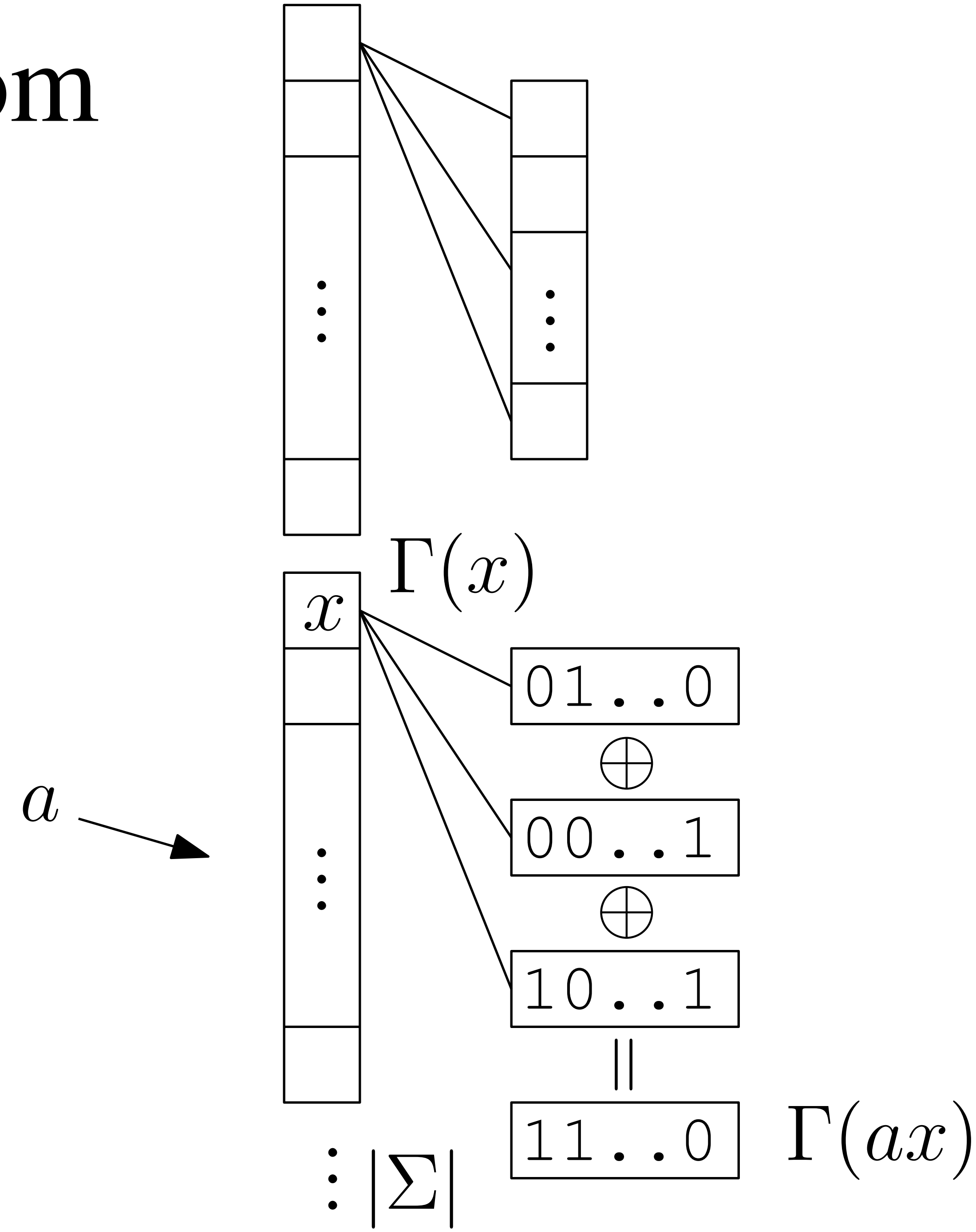

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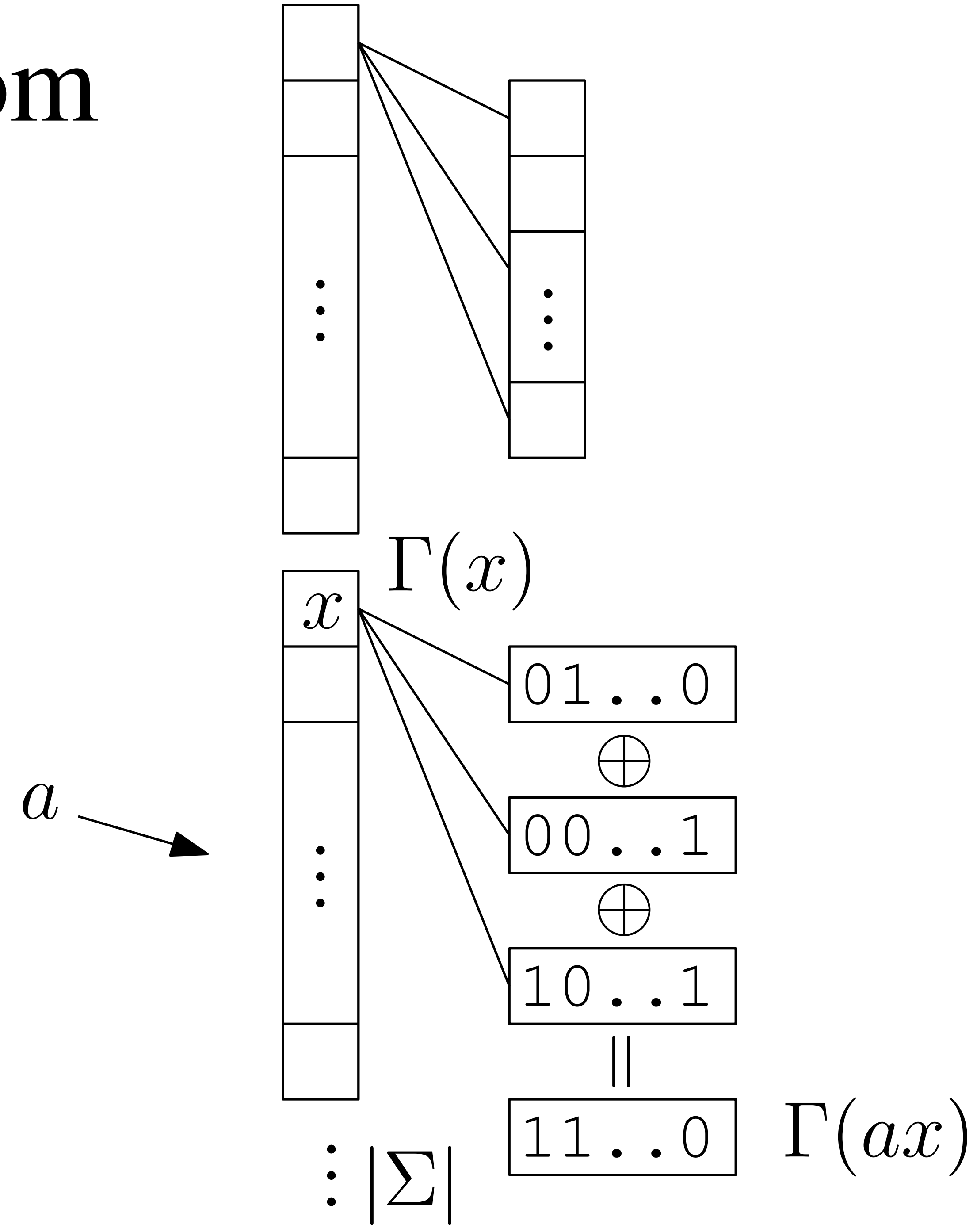
Lemma Γ is k-unique over Σ^j
$\Rightarrow \Gamma$ is k-independent over Σ^{j+1}
$\Rightarrow \Gamma$ is k-unique over Σ^{j+1} whp.



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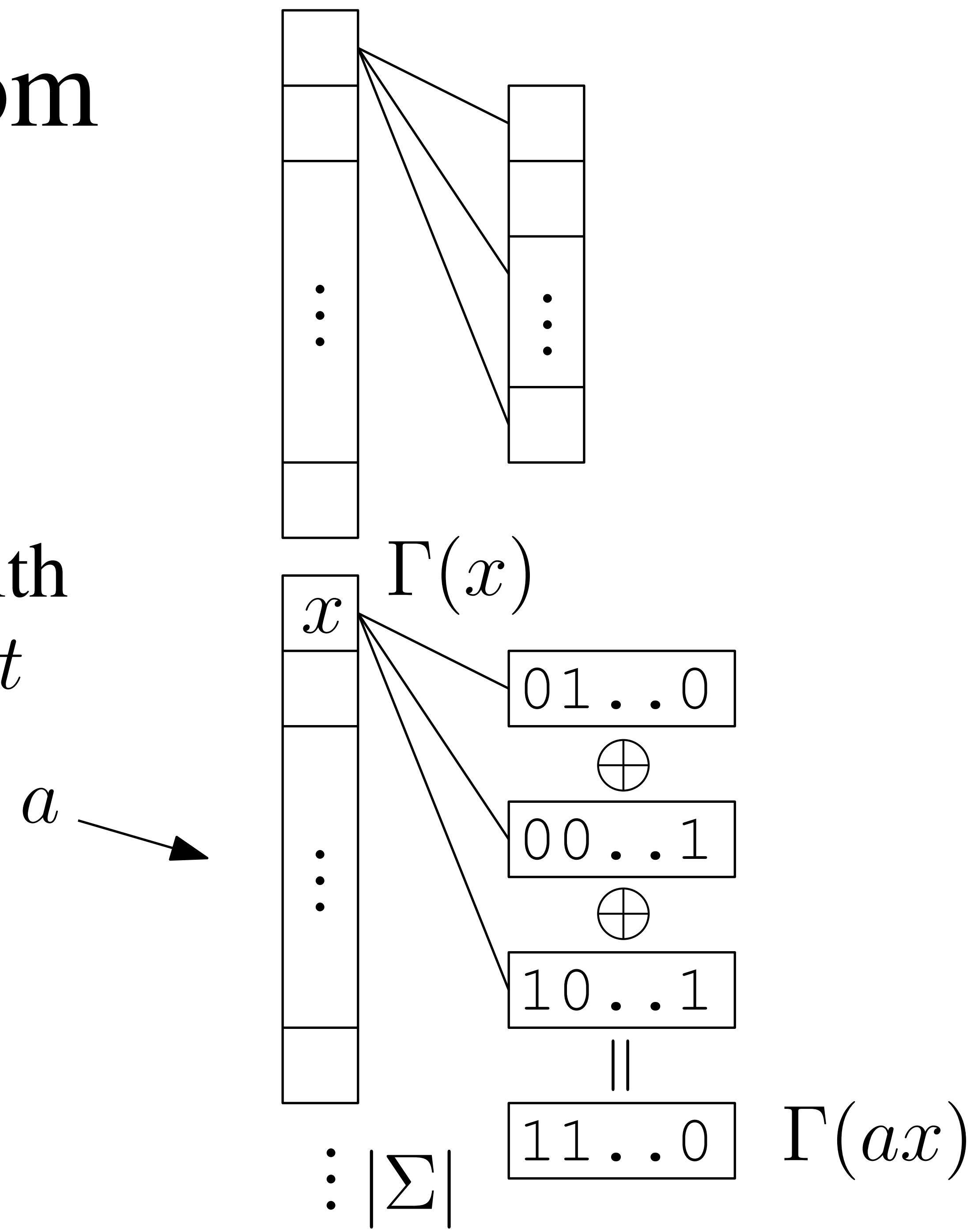
- Parameterizing for k-independence with domain size u and tradeoff parameter t

$$\begin{aligned} |\Sigma| &= u^{1/2t} \\ |V| &= O(ku^{1/2t}t) \\ d &= O(t) \end{aligned}$$

- Space $O(ku^{1/t}t^2)$. Time $O(t^2)$.



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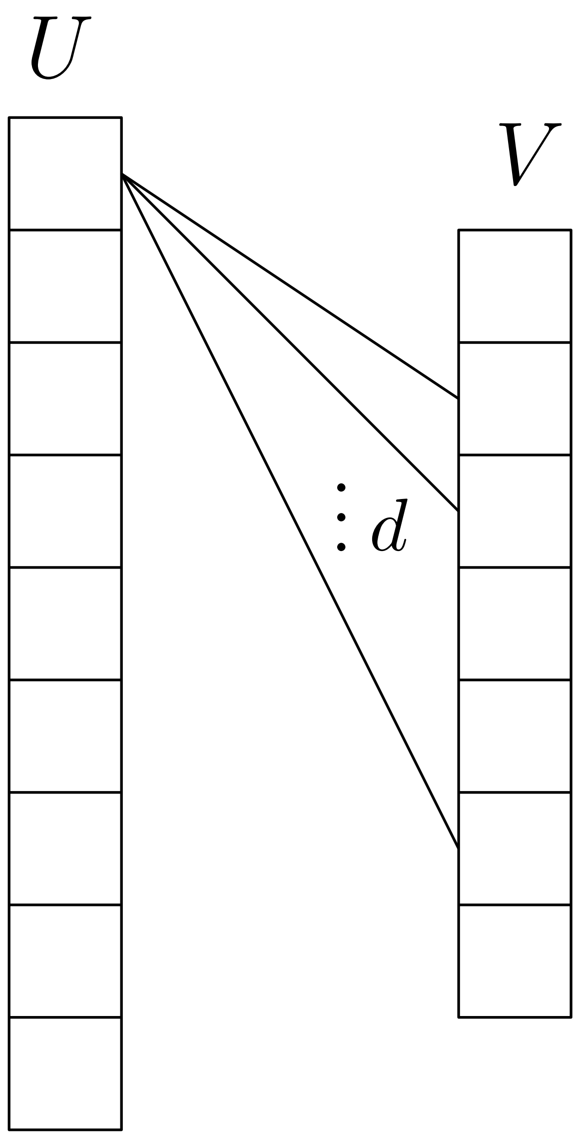

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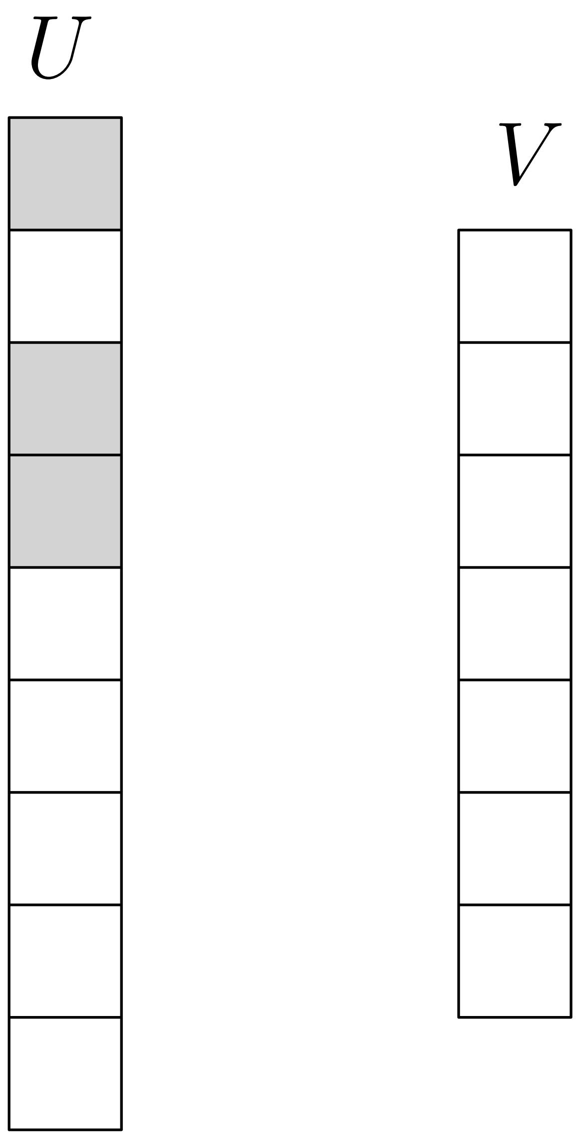

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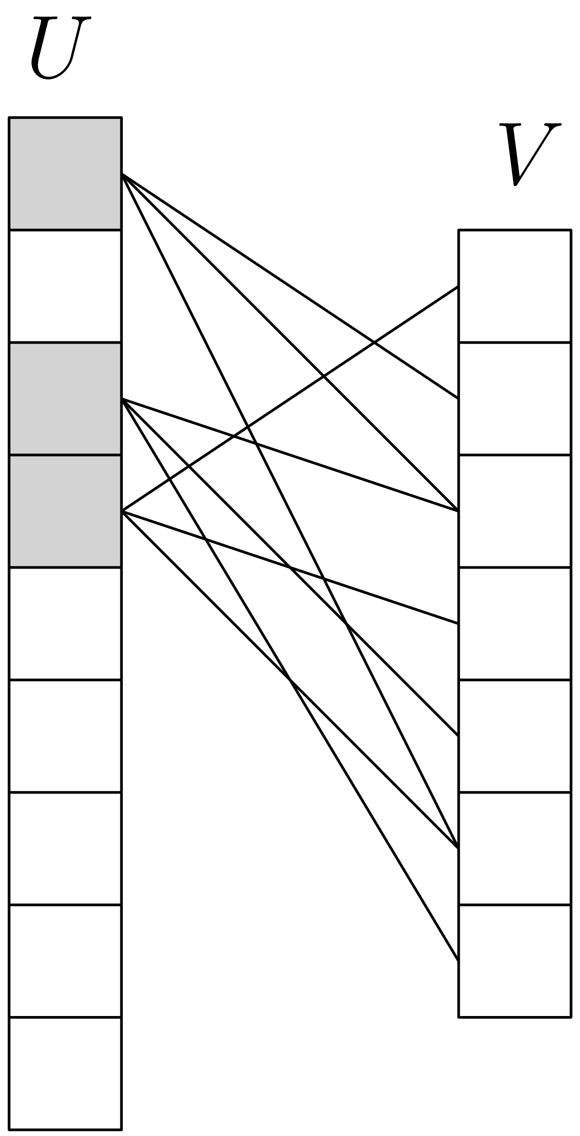

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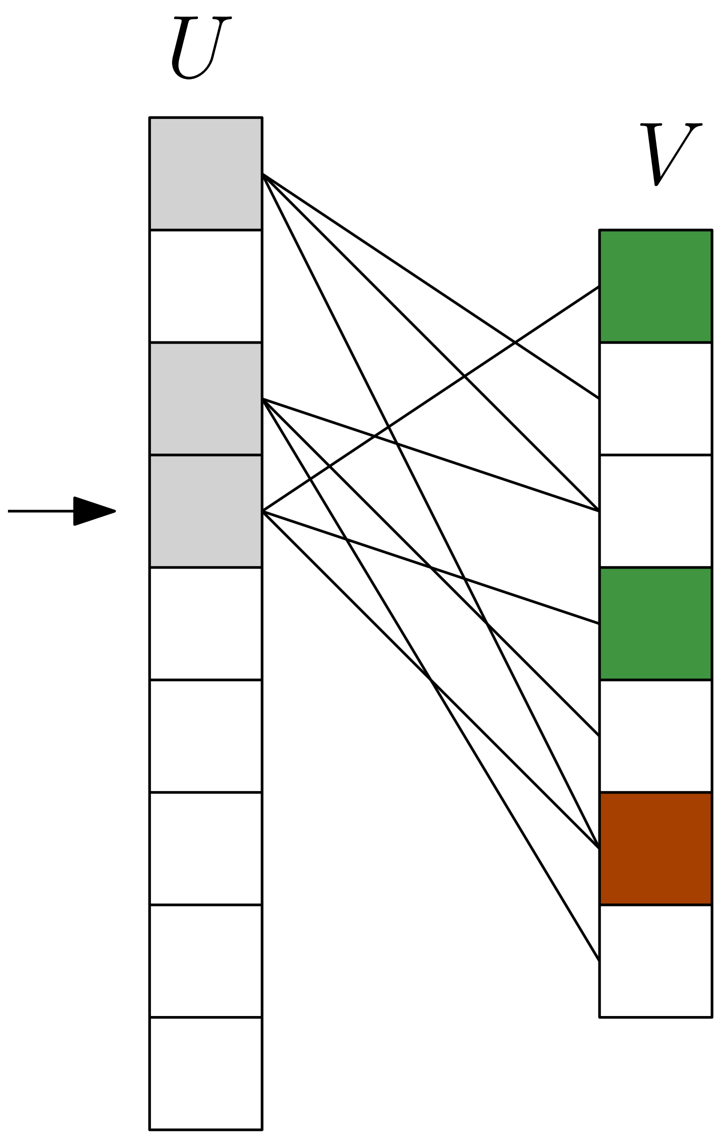

Lemma Let Γ be k-majority-unique over an alphabet Σ. Then the function defined by $\Gamma(x_1x_2)_i = \Gamma(x_1)_i\Gamma(x_2)_i$ is k-unique over Σ^2.

A graph product based on component-wise concatenation

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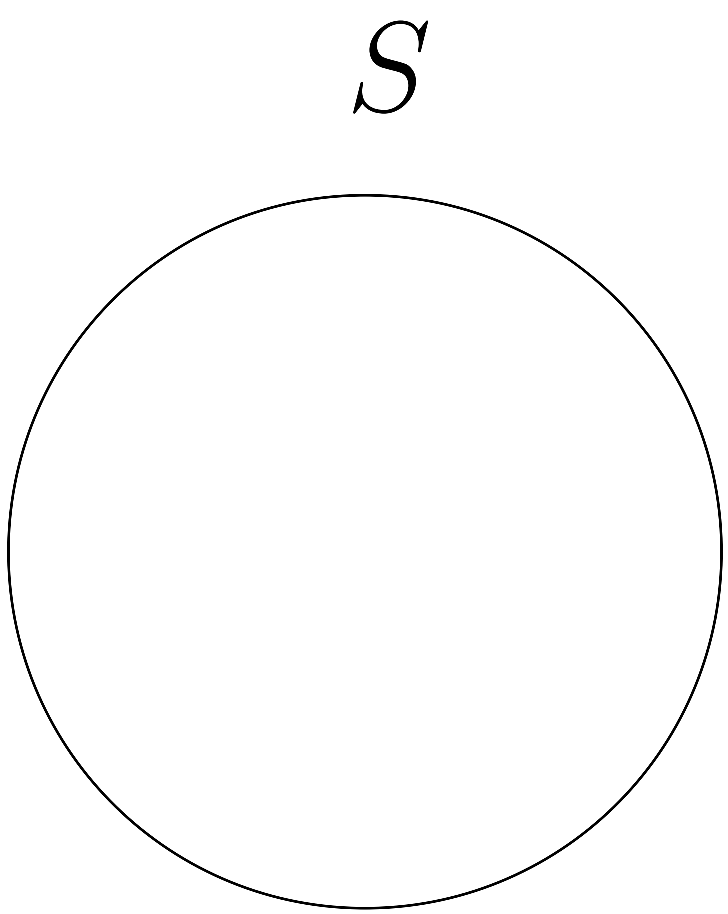

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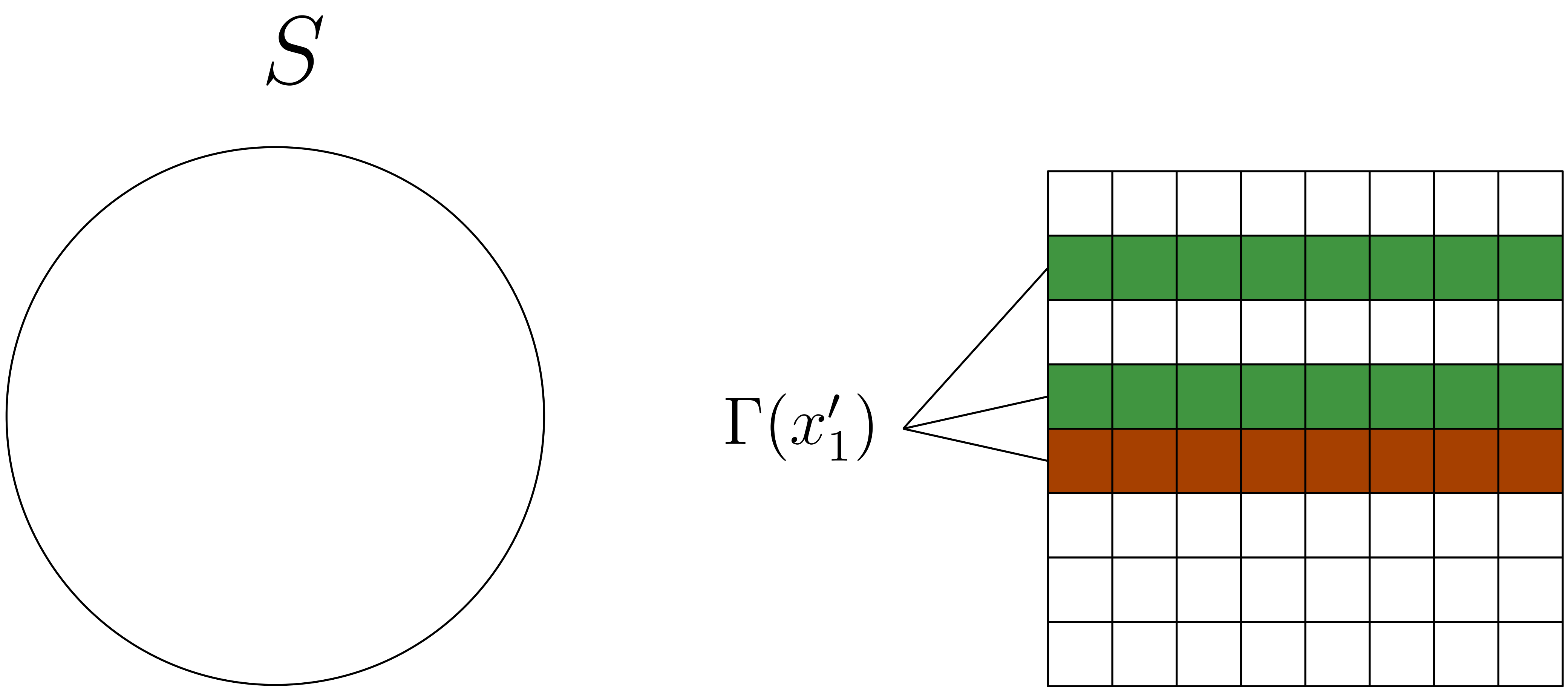

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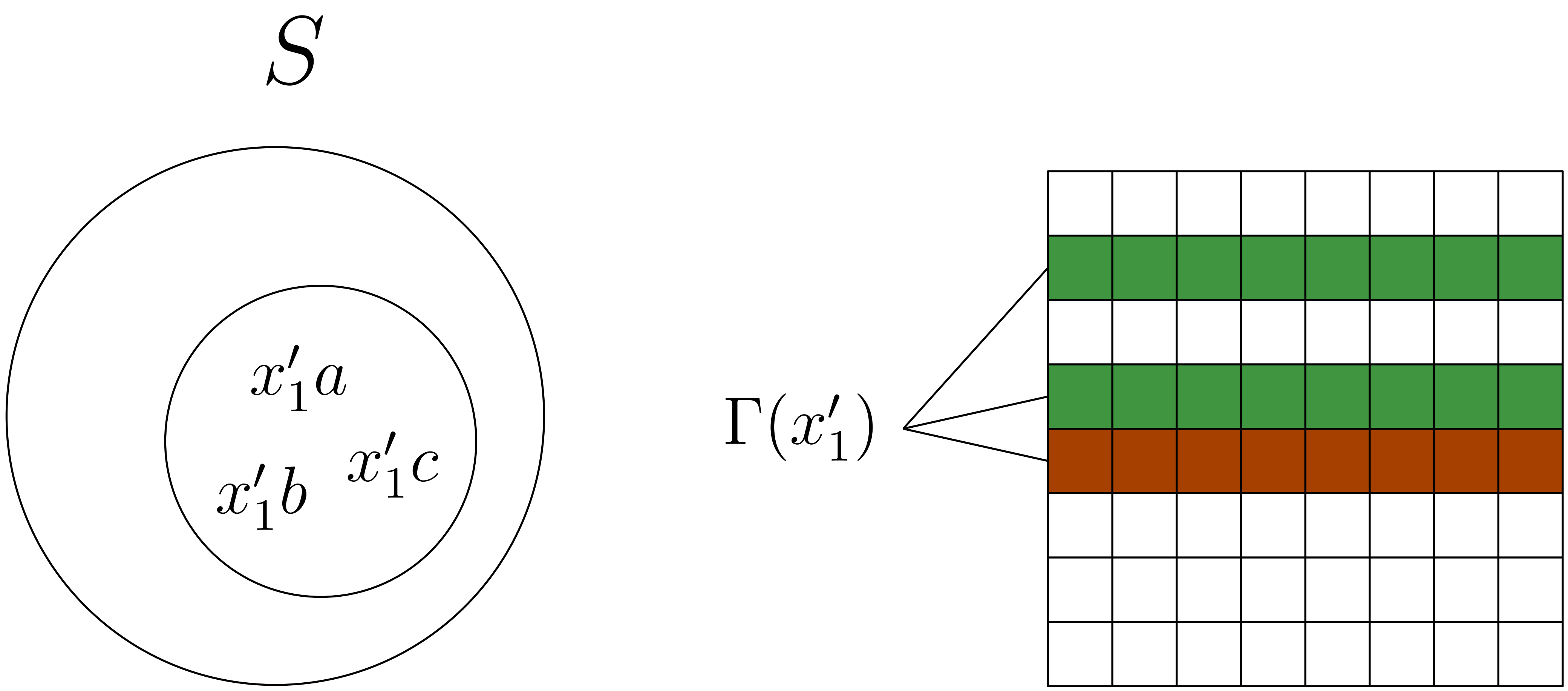

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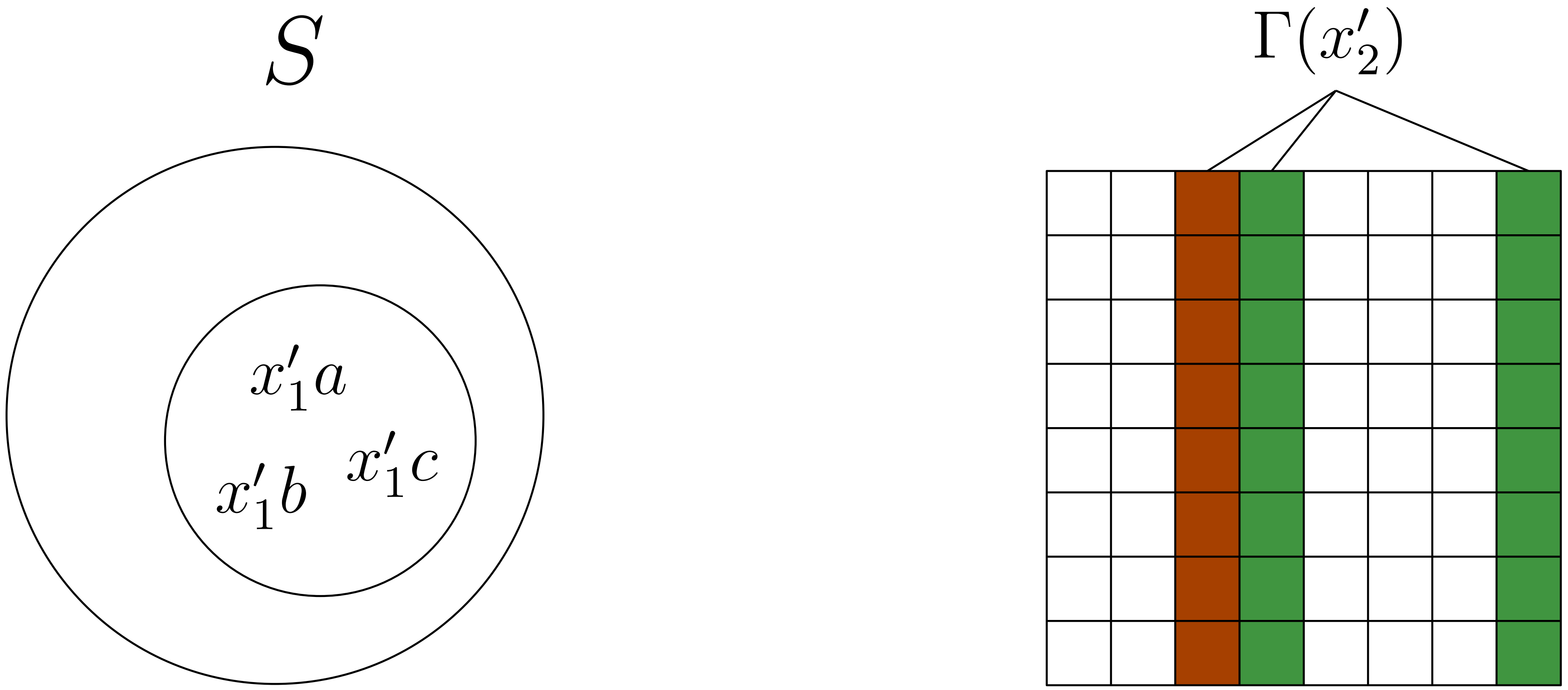

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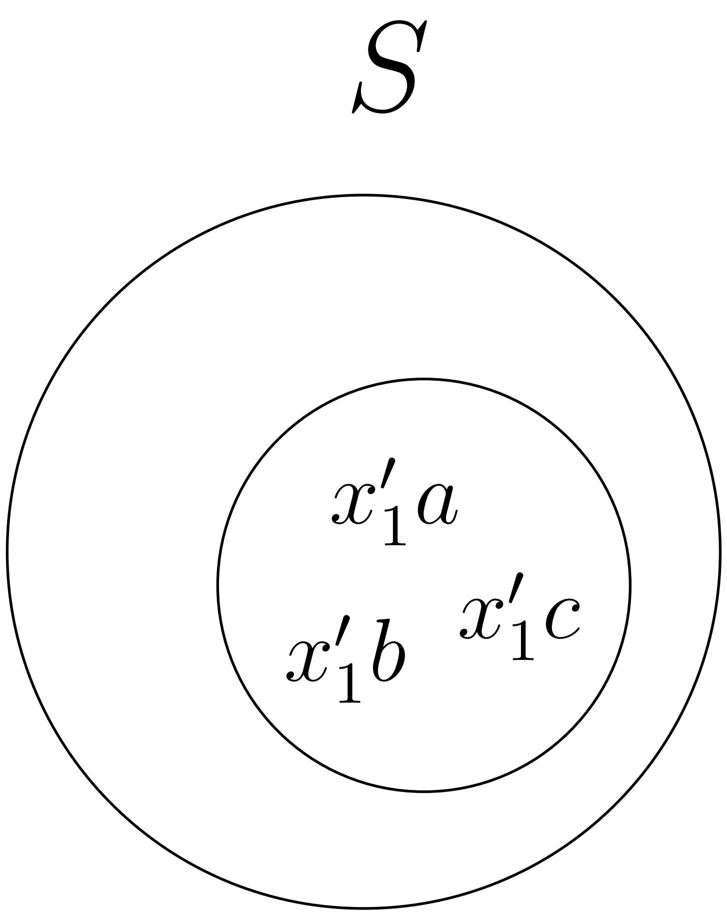

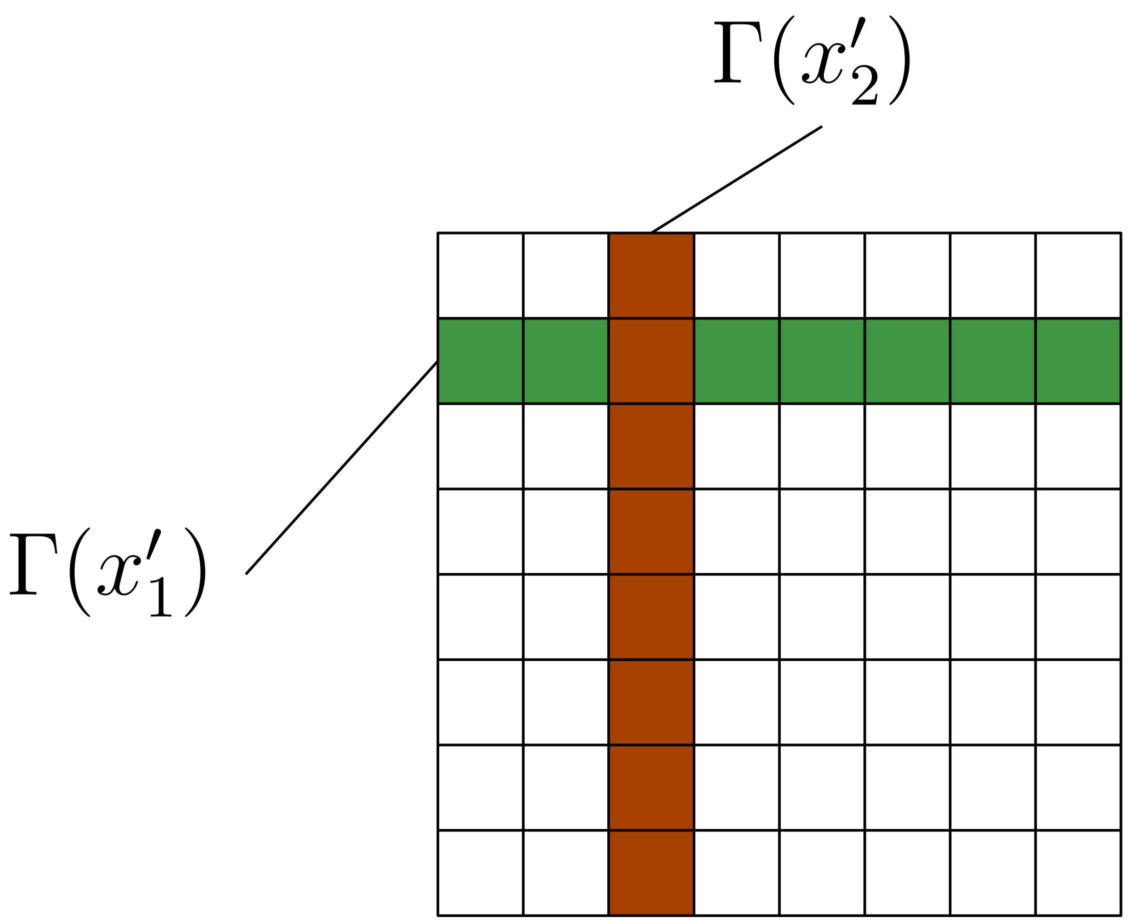

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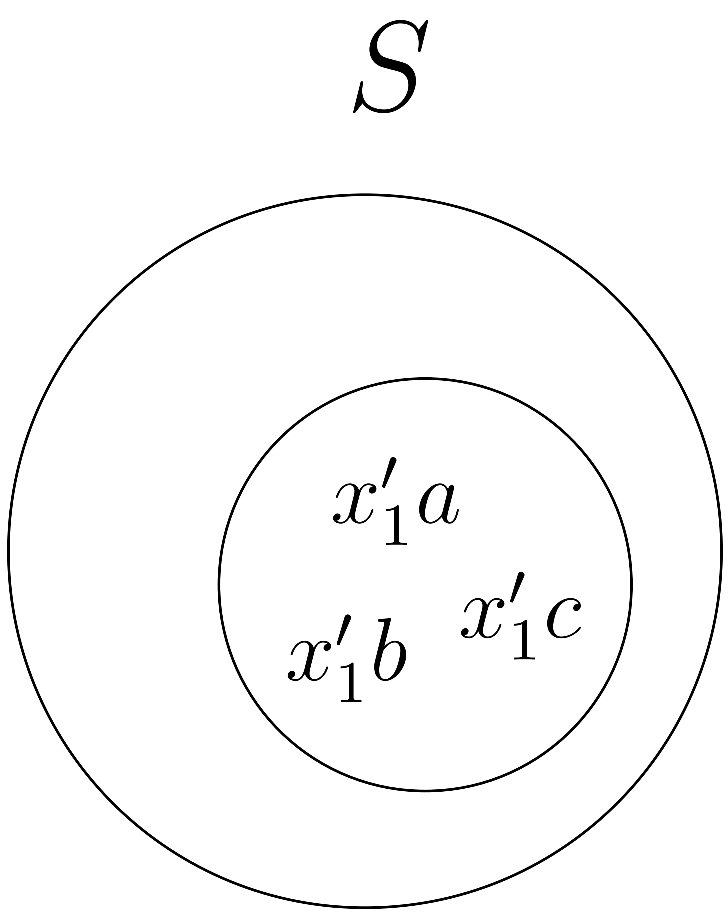

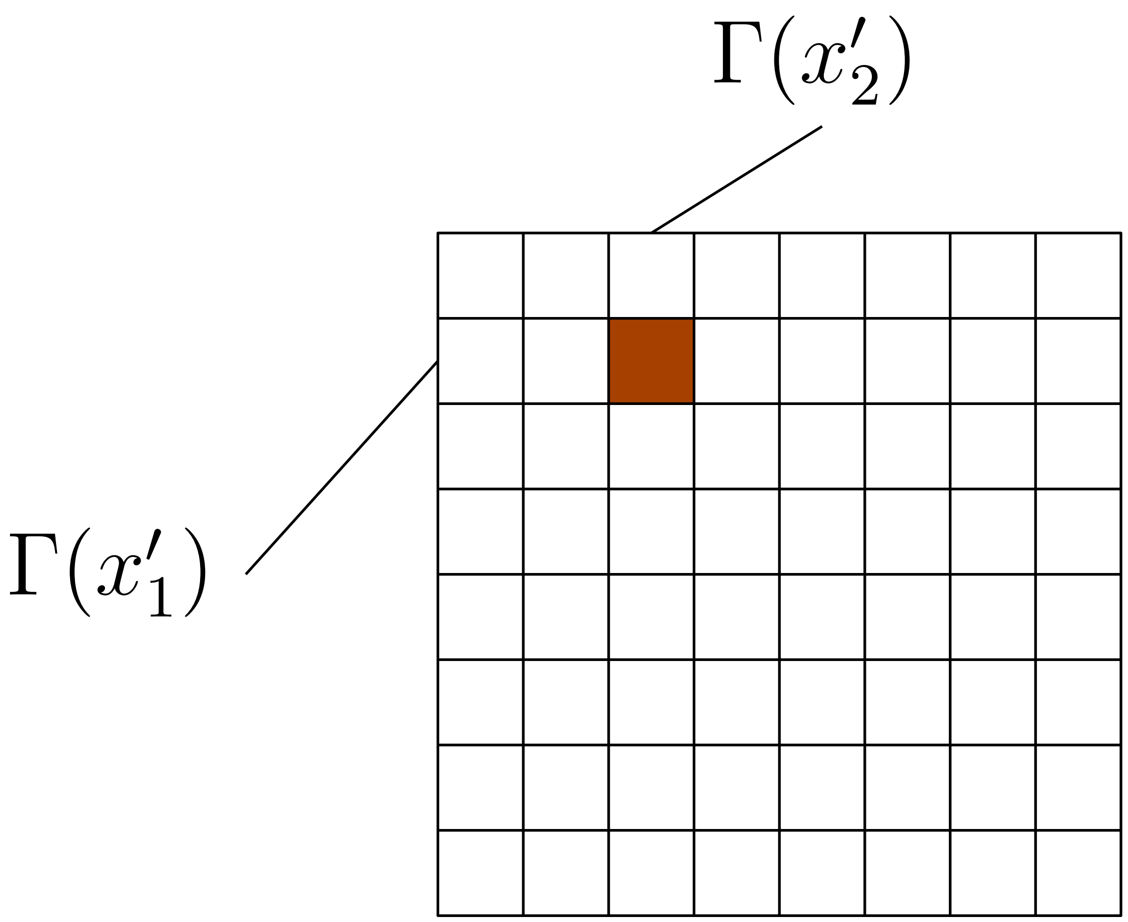

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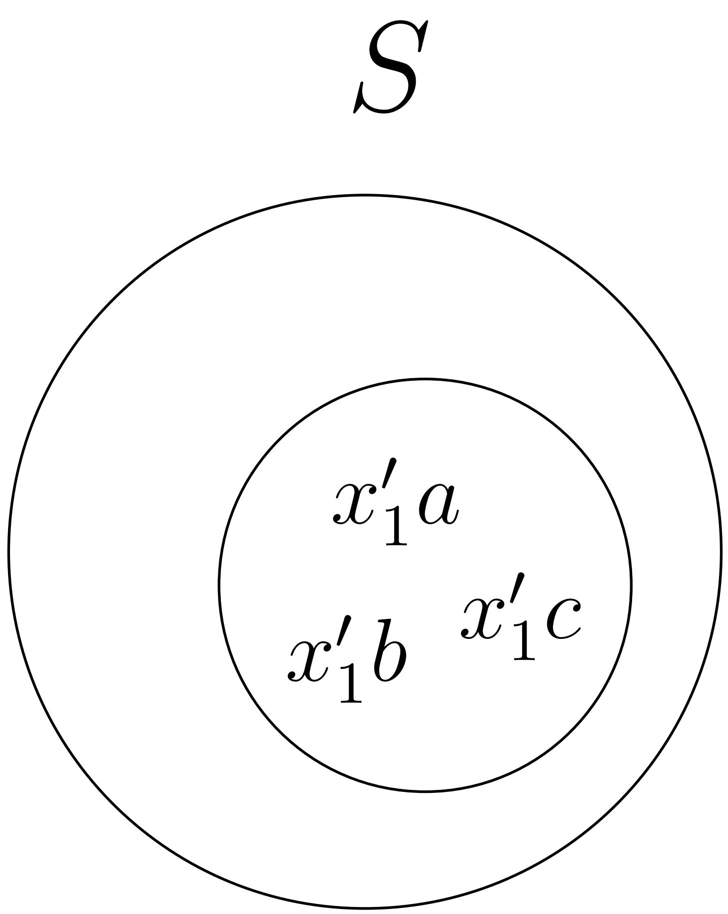

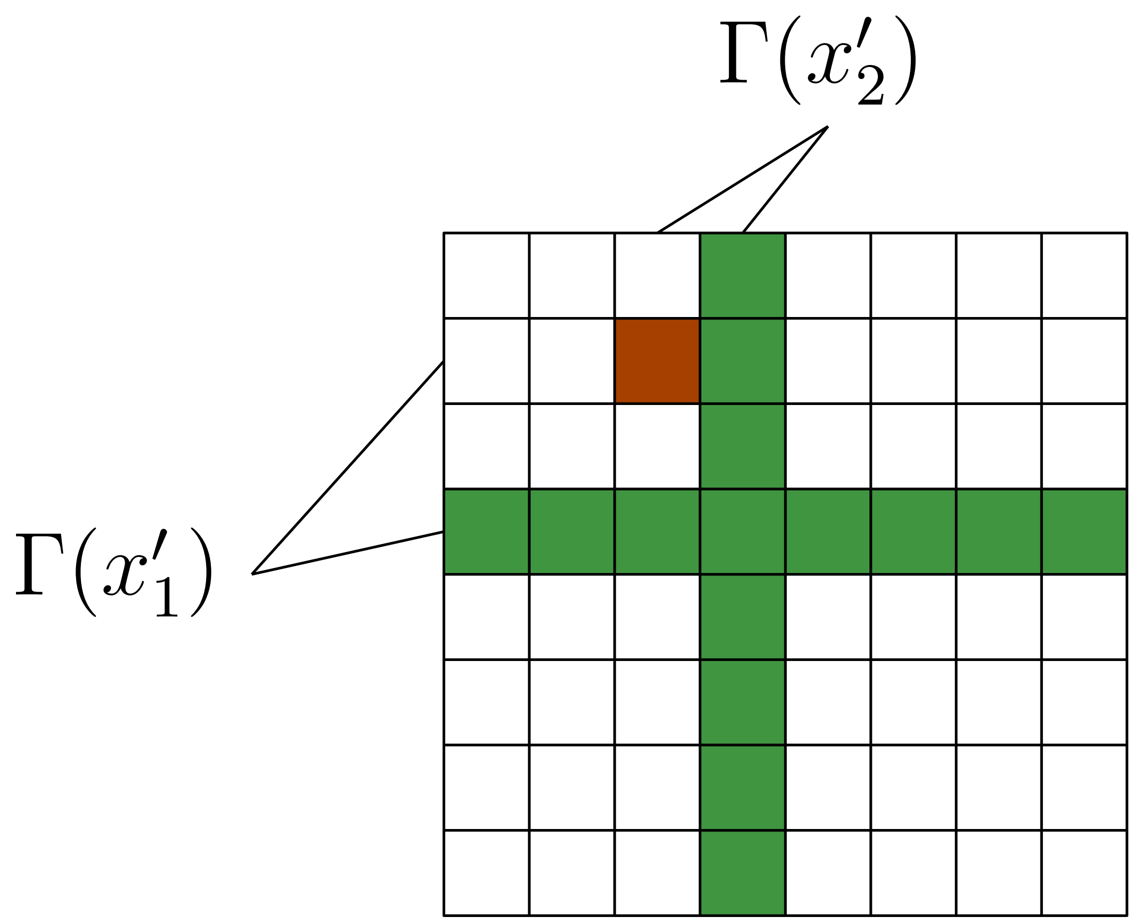

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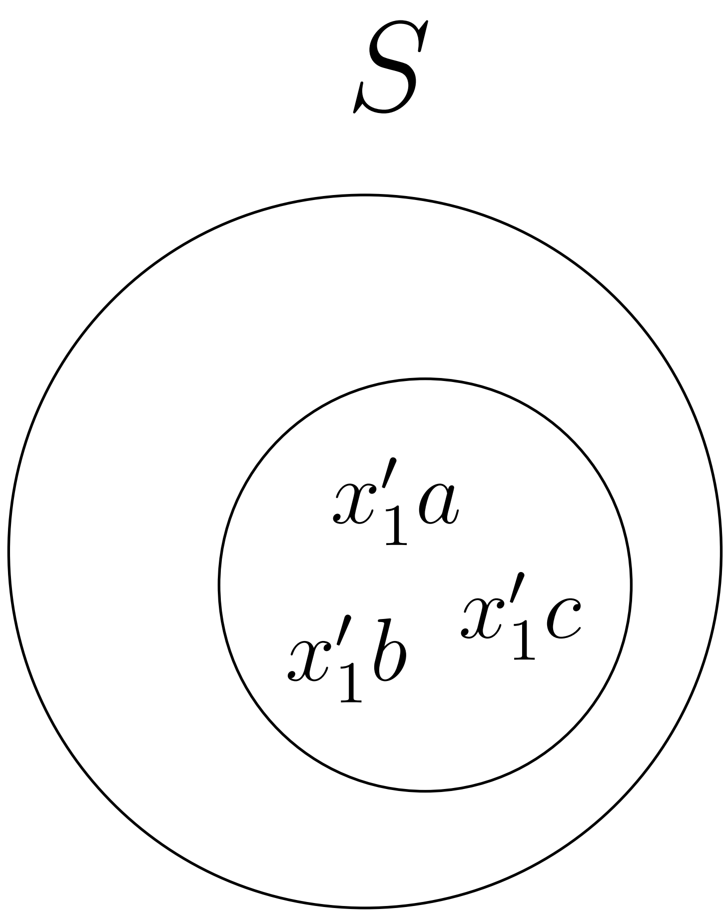

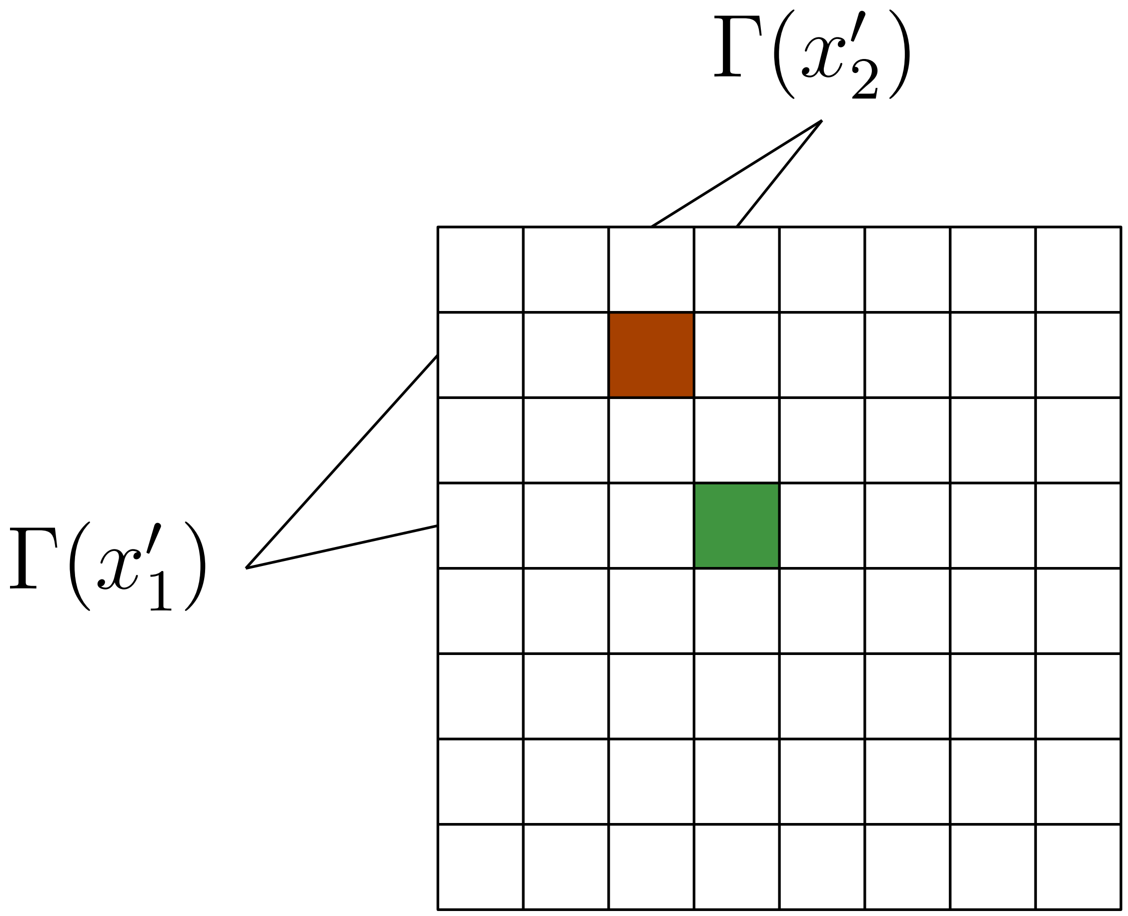

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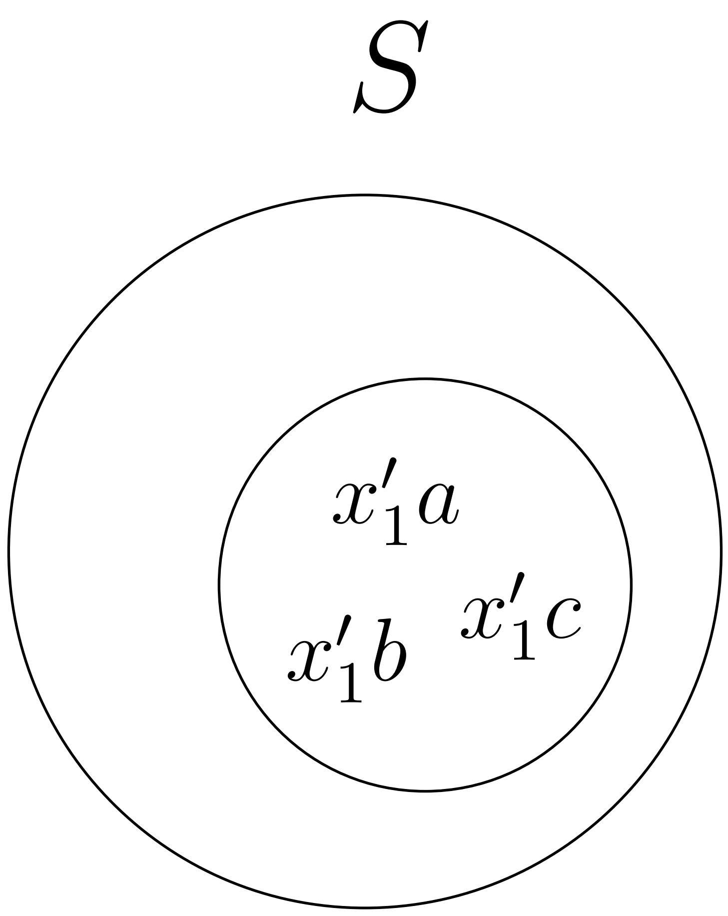

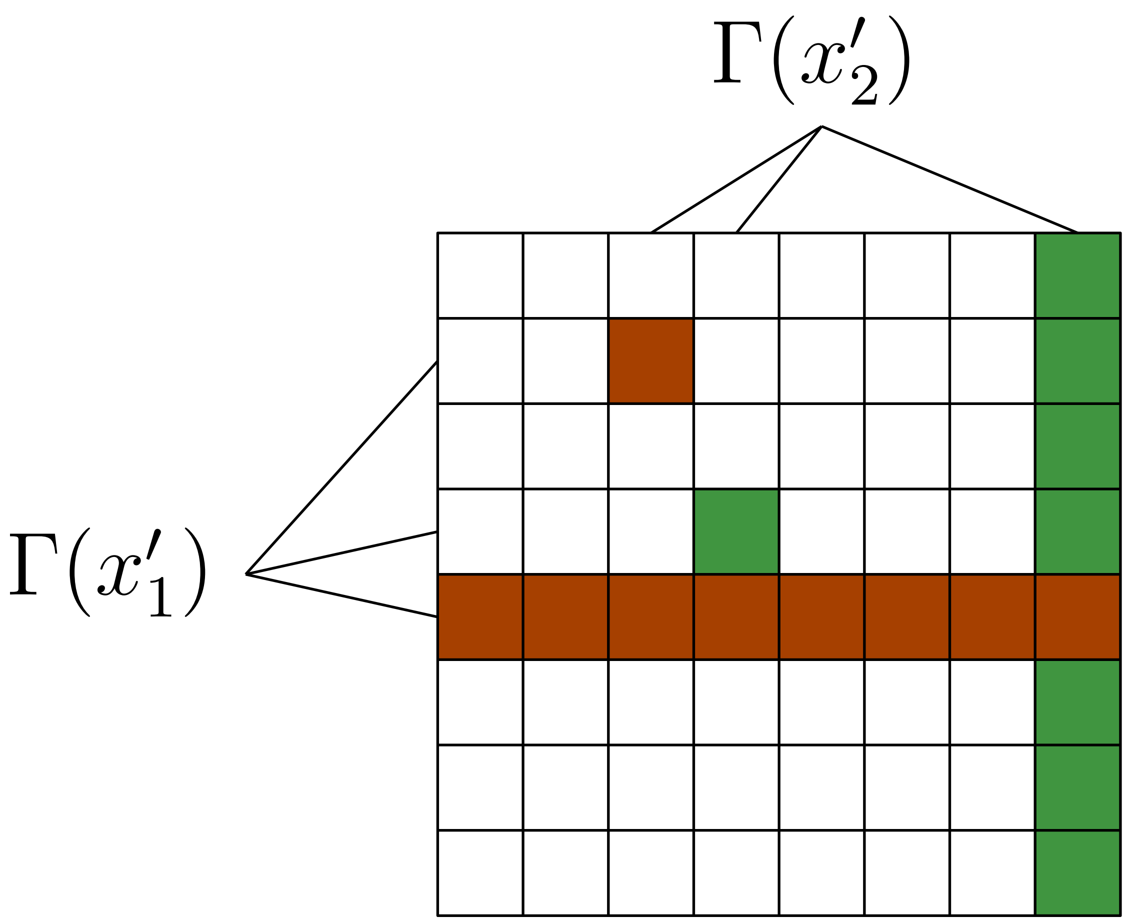

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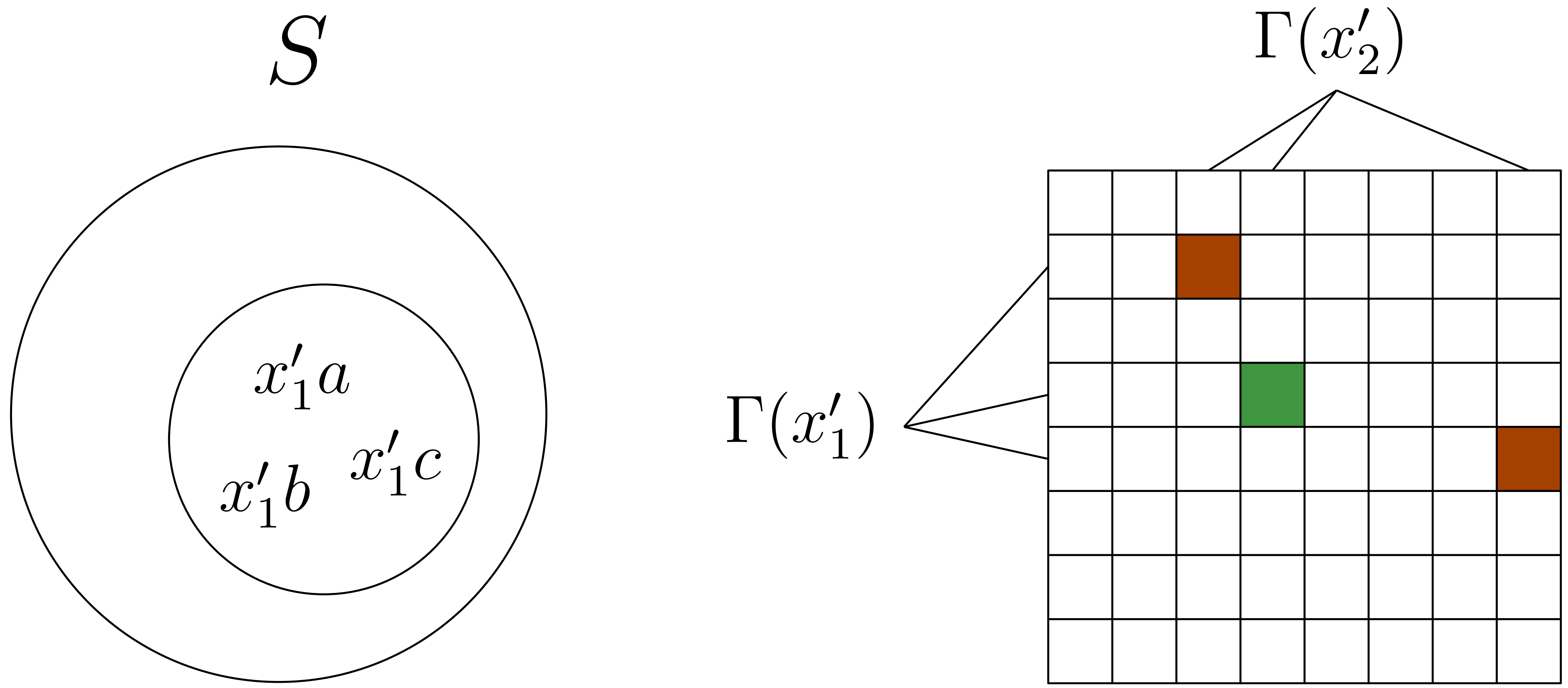

- View $x \in [u]$ as a string of two characters and recurse on each

$$\Gamma(x_1 x_2) = \bigoplus_i h(\Gamma(x_1)_i, \Gamma(x_2)_i)$$

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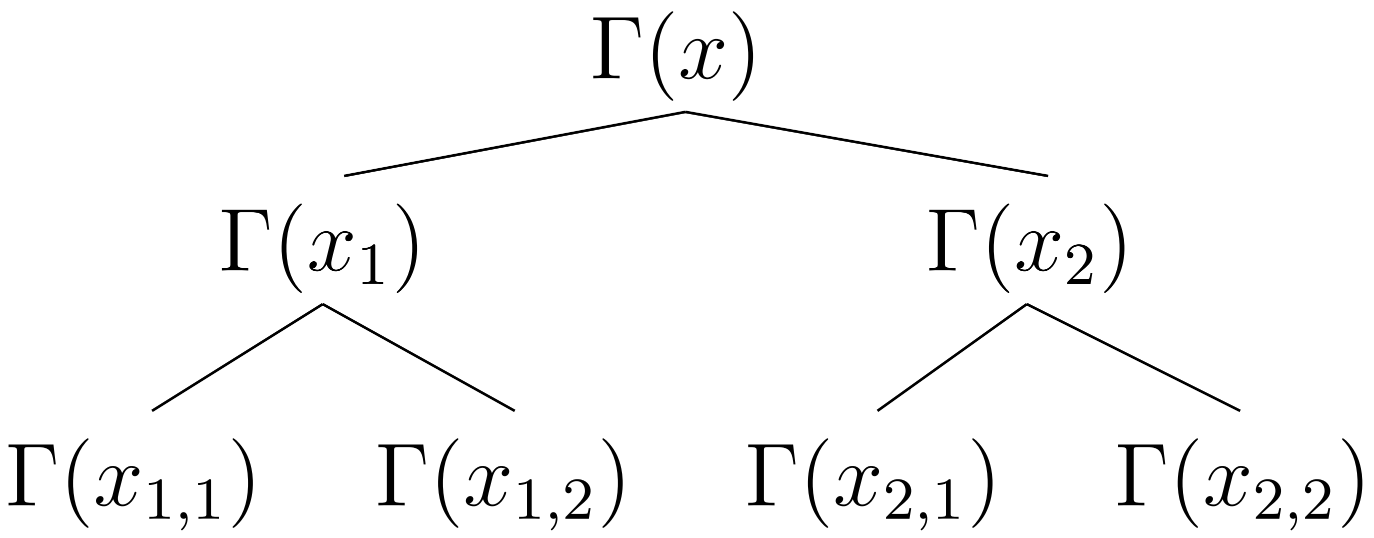

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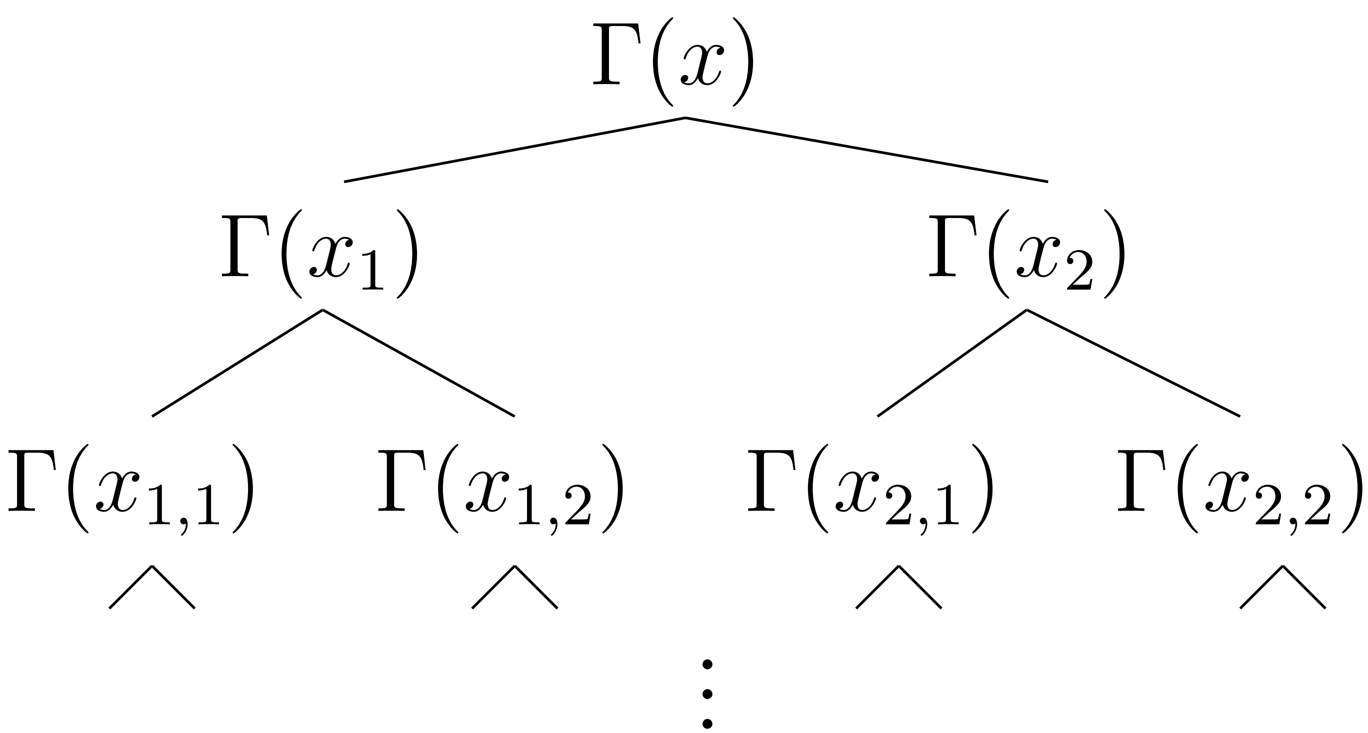

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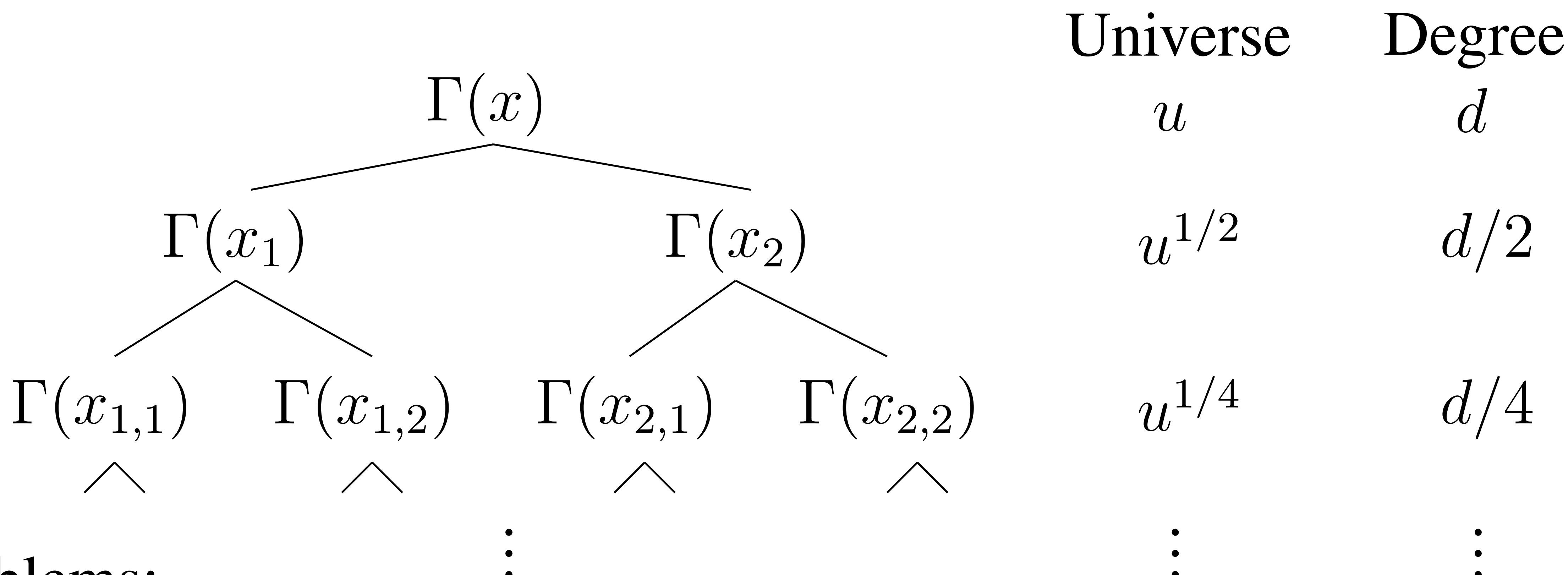

Problems:

- By the lower bound h must have a domain of size at least k^2
  - Use the first recursion to implement h

A divide-and-conquer recursion

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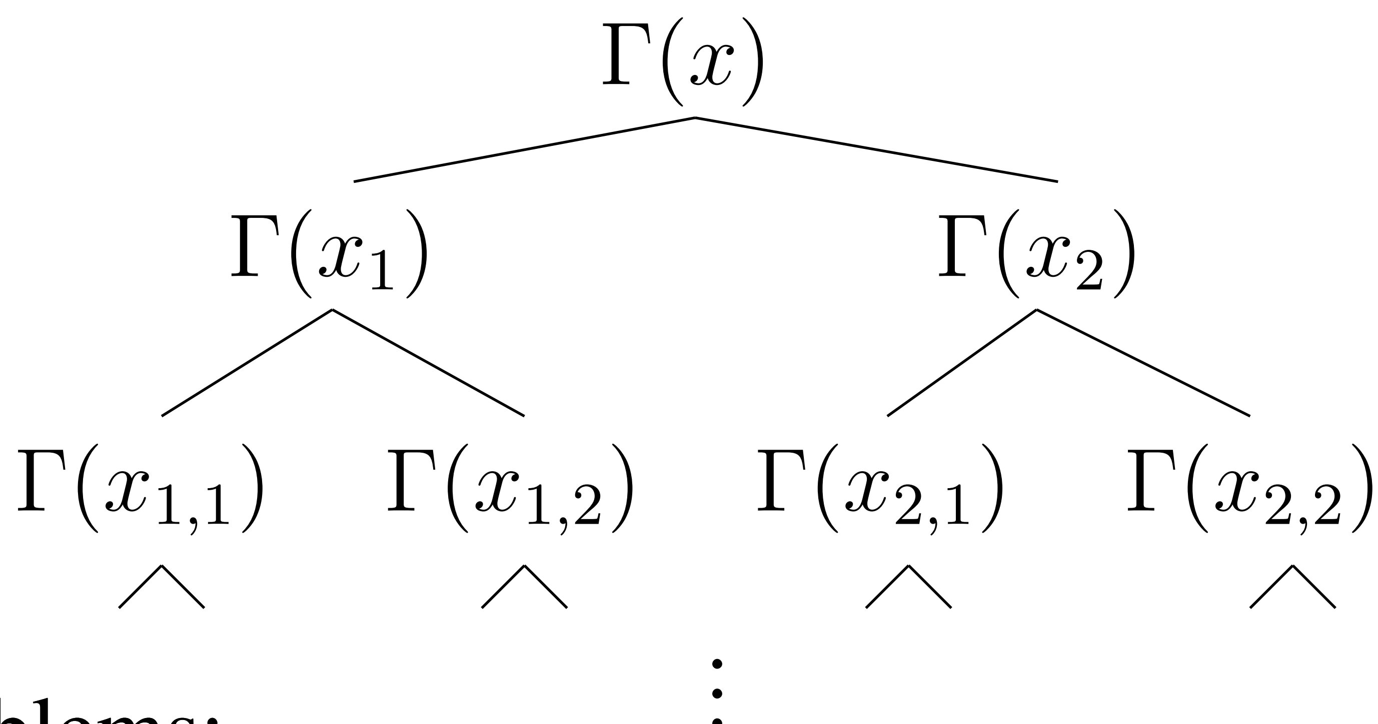

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- Low space usage, high degree: representing $\Gamma(x)$ in few words
  - Graph products that take the structure of S into account

Result:

- Near-optimal space-time tradeoff for k-independent functions

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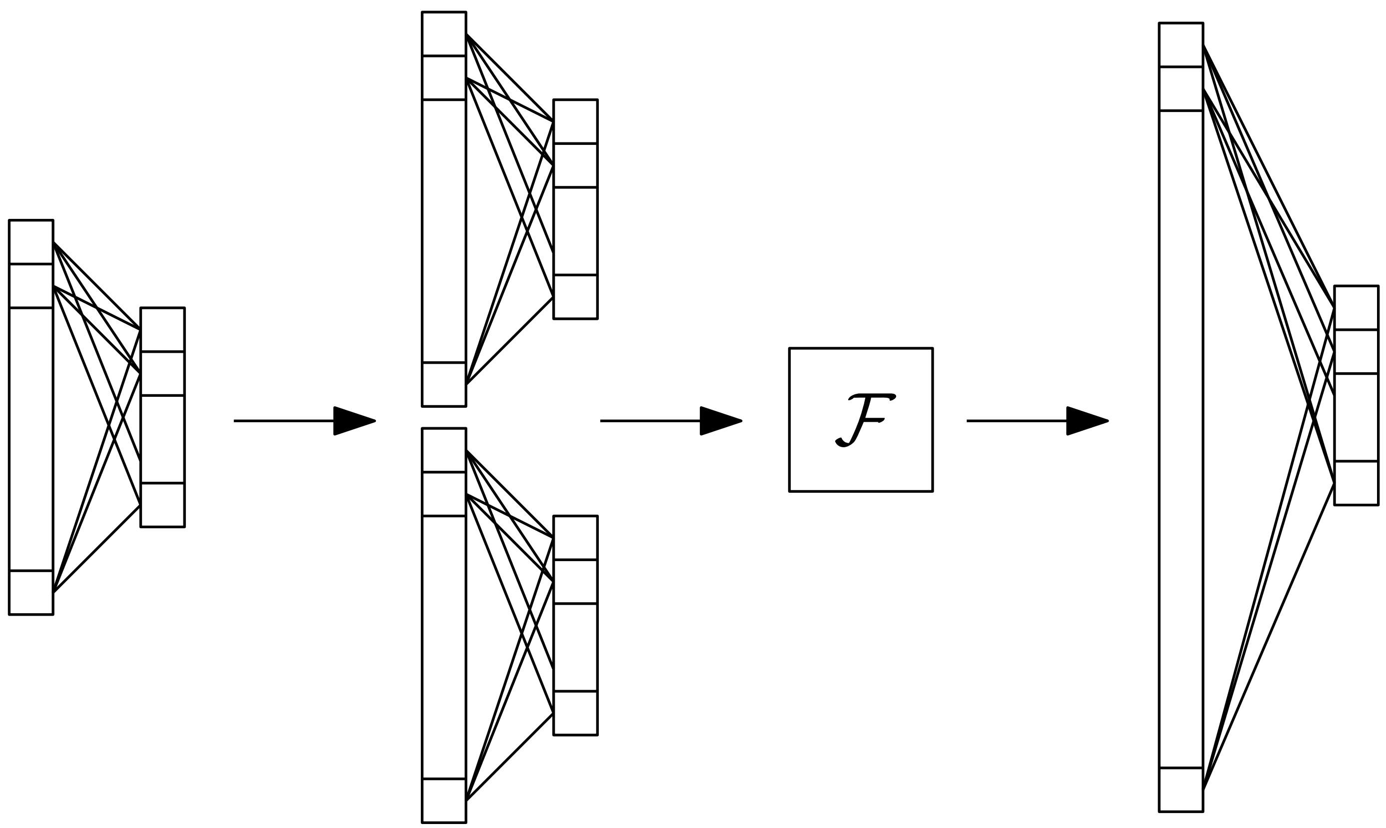
Technique:

- Graph products and alternating between expansion and independence



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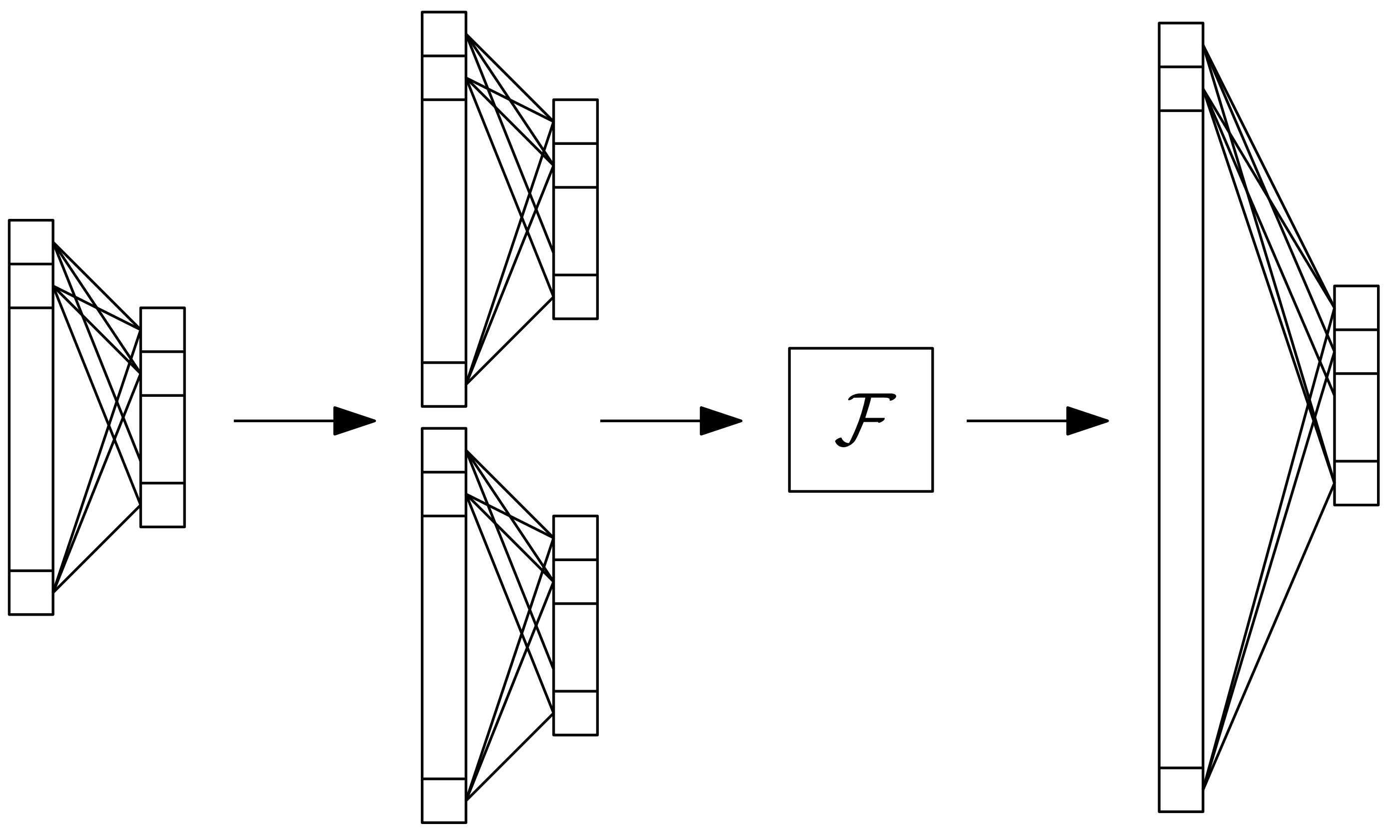

Open questions:

- Optimal expanders without k-independence

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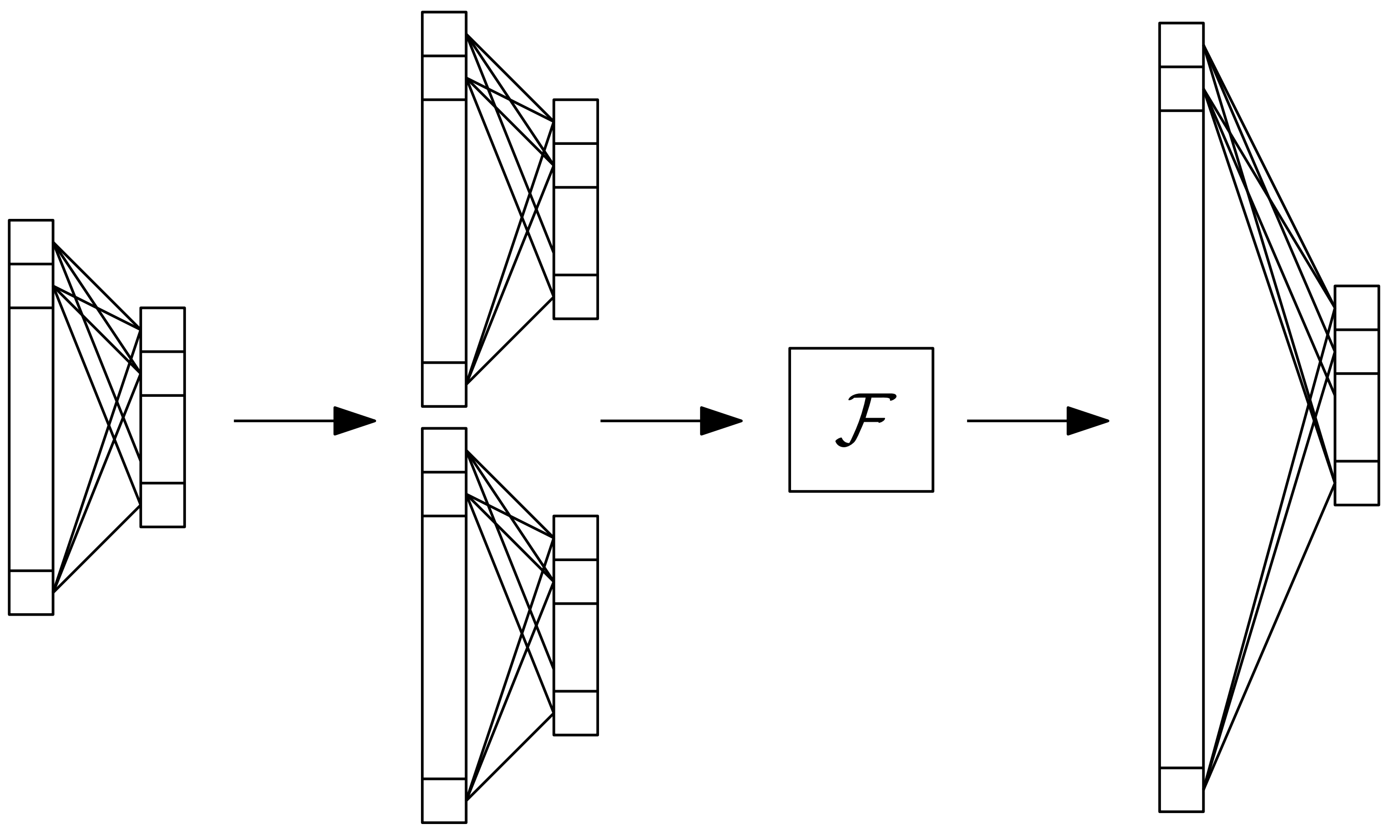

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Thanks for listening!