Discrete Mathematics
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# Homework 10

due *before* class meets.

1. Let $a_n = 2^n + 5 \cdot 3^n$ for $0 \leq n$.
   (a) Find $a_0, a_1, a_2, a_3, a_4$.
   (b) Show that $a_2 = 5 \cdot a_1 - 6 \cdot a_0$, $a_3 = 5 \cdot a_2 - 6 \cdot a_1$, and $a_4 = 5 \cdot a_3 - 6 \cdot a_2$.
   (c) Show that $a_n = 5 \cdot a_{n-1} - 6 \cdot a_{n-2}$ for all integers $2 \leq n$.

2. Find the solution to each of theses recurrence relations with the given initial conditions. Use an iterative approach such as that used in class to come up with your conjecture. Verify your conjecture.
   (a) $a_n = -a_{n-1}, a_0 = 5$.
   (b) $a_n = a_{n-1} + 3, a_0 = 1$.
   (c) $a_n = a_{n-1} - n, a_0 = 4$.
   (d) $a_n = 2a_{n-1} - 3, a_0 = -1$.
   (e) $a_n = (n+1)a_{n-1}, a_0 = 2$.
   (f) $a_n = 2na_{n-1}, a_0 = 3$.
   (g) $a_n = -a_{n-1} + n - 1, a_0 = 7$.

3. Show that
   (a) $(x^2 + 1)/(x + 1)$ is $O(x)$.
   (b) $(x^2 + 1)/(x + 1)$ is $O(x)$.
   (c) $x^3$ is $O(x^4)$.
   (d) $x^4$ is not $O(x^3)$.
   (e) $x \log x$ is $O(x^2)$.

4. Give as good a big-O estimate as possible for each of these functions.
   (a) $(n^2 + 8)(n + 1)$.
   (b) $(n \log n + n^2)(n^3 + 2)$.
   (c) $(n! + 2^n)(n^3 + \log(n^2 + 1))$.

5. Suppose that in a variation of the game of nim we allow a player to either remove one or more stones from a pile or merge the stones from two piles into one pile as long as at least one stone remains. Draw the game tree for this variation of nim if the starting position consists of three piles containing two,two, and one stone, respectively. Find the values of each vertex in the game tree and determine the winner if both players follow an optimal strategy.