SGDM E2011: Discrete Mathematics
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Final Examination

Instructions

- This is a open book examination.
- There are 4 pages.
- This examination consists of 6 questions worth 100 points. The point value of each question is given with the question.
- Read each question completely before attempting to solve any part.
- Write your answers legibly.

1 Logic [15 points]

Recall from class the following rules defining conjunction, disjunction, and implication.

$$\frac{A \quad B}{A \wedge B}\,\wedge\mathrm{I} \qquad \frac{A \wedge B}{A}\,\wedge\mathrm{E}_1 \qquad \frac{A \wedge B}{B}\,\wedge\mathrm{E}_2$$

$$\frac{\begin{array}{c}\overline{A}^{\,u}\\ \vdots \\ B\end{array}}{A \Rightarrow B}\,\Rightarrow\mathrm{I}^u \qquad \frac{A \Rightarrow B \quad A}{B}\,\Rightarrow\mathrm{E}$$

$$\frac{A}{A \vee B}\,\vee\mathrm{I}_1 \qquad \frac{B}{A \vee B}\,\vee\mathrm{I}_2 \qquad \frac{A \vee B \quad \begin{array}{c}\overline{A}^{\,u}\\ \vdots \\ C\end{array} \quad \begin{array}{c}\overline{B}^{\,v}\\ \vdots \\ C\end{array}}{C}\,\vee\mathrm{E}^{u,v}$$

Questions 1.1 [5 points] Give a derivation of

$$(A \wedge B) \Rightarrow A$$

$$\frac{\dfrac{\overline{A \wedge B}^{\,u}}{A}\,\wedge E_1}{A \wedge B \Rightarrow A}\,\Rightarrow I^u$$

Questions 1.2 [10 points] Give a derivation of

$$(A \Rightarrow C) \wedge (B \Rightarrow D) \Rightarrow (A \vee B) \Rightarrow (C \vee D)$$

$$
\dfrac{\dfrac{\dfrac{\overline{A \vee B}^{\,v} \qquad \dfrac{\dfrac{\dfrac{\overline{F}^{\,u}}{A \Rightarrow C}\,\wedge E_1 \qquad \overline{A}^{\,w_1}}{C}\,\Rightarrow E}{C \vee D}\,\vee I_1 \qquad \dfrac{\dfrac{\dfrac{\overline{F}^{\,u}}{C \Rightarrow D}\,\wedge E_2 \qquad \overline{B}^{\,w_2}}{D}\,\Rightarrow E}{C \vee D}\,\vee I_2}{C \vee D}\,\vee E^{w_1,w_2}}{(A \vee B) \Rightarrow (C \vee D)}\,\Rightarrow I^{v}}{\underbrace{(A \Rightarrow C) \wedge (B \Rightarrow D)}_{F} \Rightarrow (A \vee B) \Rightarrow (C \vee D)}\,\Rightarrow I^{u}
$$

2 Induction [15 points]

Consider the the function $f(n) = n^3 - n$ for $n \geq 2$. Recall that "n is divisible by m" is defined as that there exists a k, such that $n = m \cdot k$.

Question 2.1 [5 points] Let's get a feel for this function. Compute $f(n)$ for the first five number, i.e. $n = 2, 3, 4, 5, 6$.

$f(2) = 2^3 - 2 = 8 - 2 = 6 \quad = 2 \cdot 3$

$f(3) = 3^3 - 3 = 27 - 3 = 24 \quad = 8 \cdot 3$

$f(4) = 4^3 - 4 = 64 - 4 = 60 \quad = 20 \cdot 3$

$f(5) = 5^3 - 5 = 125 - 5 = 120 \quad = 40 \cdot 3$

$f(6) = 6^3 - 6 = 36 \cdot 6 - 6 = 216 = 210 \quad = 70 \cdot 3$

Question 2.2 [10 points] From Question 2.1., we may come up with the hypothesis that $f(n)$ is always divisible by 3. This is what you will do: Prove formally that $f(n)$ is divisible by 3 by induction for all $n \geq 2$. *Hint: Prove base case and step case, and write out clearly the induction hypothesis.*

Base Case: $n = 2$

already shown: $f(2) = 6 = 2 \cdot 3$ ✓

Step case: Let m be arbitrary but fixed

Induction Hypothesis: $m^3 - m$ is divisible by 3

To show: $f(m+1)$ is divisible by 3

$$f(m+1) = (m+1)^3 - (m+1) = (m+1)((m+1)^2 - 1) = (m+1)(m^2 + 2m \cancel{+1} \cancel{-1})$$

$$= m^3 + 2m^2 + m^2 + 2m = m^3 + 3m^2 + 2m + m - m = (m^3 - m) + 3(m^2 + m)$$

$$= 3 \cdot k + 3(m^2 + m) = 3(k + m^2 + m)$$

which is the very definition of divisibility by 3!

this k is known to exist. because $m^3 - m$ is divisible by 3 by the induction hypothesis. $\square$

3 Complexity [15 points]

Consider the function

$$T(n) = n^4 + 3n^3 + 4n + 15$$

You might think of this function as the number of steps necessary to compute some output from an input of size n.

Question 3.1 [5 points] Show that $T(n) \in O(n^4)$.

We must show that there is a c, such that for all $n >$ some k_0

$$T(n) = n^4 + 3n^3 + 4n + 15 < c\, n^4$$

$$n^4 + 3n^3 + 4n + 15 < n^4 + 3n^4 + 4n^4 + 15n^4$$

$$= 23n^4$$

for $n > 1$

Thus choose $k_0 = 1$ and $c = 23$!

$\square$

Question 3.1 [5 points] Show that $T(n) \notin O(n^3)$.

Assume $T(n) \in O(n^3)$

Then there is a c and k_0, such that $T(n) < cn^3$ for all $n > k_0$

$n^4 + 3n^3 + 4n + 15 < cn^3$

but this means that $n + 3 + \frac{4}{n^2} + \frac{15}{n^3} < c$.

But such a c cannot exist. Assume it does.

Choose $n = c$

Let's check:

$c + 3 + \frac{4}{c^2} + \frac{15}{c^3} < c$ iff $3 < 3 + \frac{4}{c^2} + \frac{15}{c^3} < 0$

which is impossible.

Therefore $\neg T(n) \in O(n^3)$ $\square$

Question 3.2 [5 points] Show that $T(n) \in O(n^5)$.

This problem can be solved similar to 3.1.

$T(n) < 24n^5$ for all $n > 1$

$\square$

4 Regular Expressions [20 points]

Recall from class the definition of regular expressions: $R ::= 1 \mid 0 \mid c \mid R_1 + R_2 \mid R_1 \times R_2 \mid R^*$. Consider the language $\mathcal{L}$ of strings over the alphabet $\{a, b, c\}$ that contains the sequence *abc* as a substring. A *string* is a word, a sequence of characters. For example the strings *ababca*, *aabbabcbb*, or *abcc* are contained in the language $\mathcal{L}$. The strings *abbc* and *aabac* are not.

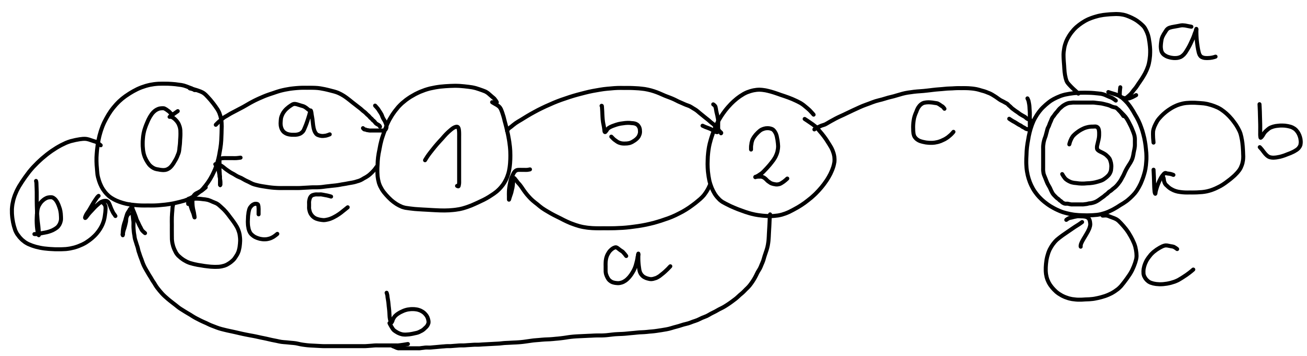
Question 4.1 [5 points] Write out a regular expression that accepts all strings over the alphabet $\{a, b, c\}$.

$R_u = (a+b+c)^*$

Question 4.2 [5 points] Give a regular expression that generates the language $\mathcal{L}$.

$$R = (R_u \cdot a \cdot b \cdot c \cdot R_u)$$

Question 4.3 [10 points] Define a deterministic finite automaton that accepts $\mathcal{L}$. Define clearly the states. Draw a diagram.



State 0: the last character seen was not an a

State 1: Last letter seen: a

State 2: Last two letters seen: ab

State 3: Recognized abc. This is a final state.

5 Models of Computation [15 points]

Consider the following λ-term

$$((\lambda f.\, \lambda x.\, f\ (f\ (x+3)))\ (\lambda y.\, y+2))\ 6.$$

Question 5.1 [5 points] How many redexes can you find. Highlight them.

There is one and only one redex:

$$(\lambda f. \lambda x.\, f(f(x+3)))\ (\lambda y.\, y+2)$$

$\square$

Question 5.2 [10 points] Give a reduction of this λ-expression to normal form.

$$((\lambda f.\, f(f(x+3)))\ (\lambda y.y+2))\ 6$$

$$((\lambda f.\lambda x.\, f(f(x+3)))\ (\lambda y.y+2))\ 6$$

$$\rightarrow (\lambda x.\,(\lambda y.y+2)((\lambda y.y+2)(x+3)))\ 6$$

$$\rightarrow (\lambda y.y+2)((\lambda y.y+2)(6+3))$$

$$\rightarrow (\lambda y.y+2)((\lambda y.y+2)\ 9)$$

(here is the only place you have to redices)

$$\rightarrow (\lambda y.y+2)\ (9+2)$$

$$\rightarrow (\lambda y.y+2)\ 11$$

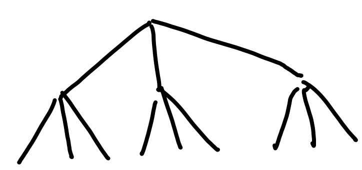
$$\rightarrow 11+2$$

$$\rightarrow 13$$

6 Trees [20 points]

Recall from class the definition of a *n-ary*: a *n-ary tree* is a tree, where every node has exactly n children. A tree is *complete* if all of its leaves are at the same depth.

Question 6.1 [5 points] How many leaves has a 3-ary tree of depth 2?



There the tree has 9 leaves. $\square$

Question 6.2 [5 points] Which one of the following options do you believe denotes the number of leaves of a complete n-ary tree of depth k. *Hint: If this is difficult for you, sketch out a few trees for different n and k, and count...*

1. $n^{2 \cdot k}$.
2. k^{n+1}.
3. k^n.
4. n^k.

With every layer of the tree, you get n times as many leaves, therefore the result must be

$$\underbrace{n \cdot n \cdots n}_{k \cdot \text{times}} = n^k$$

Therefore 4. is the correct answer. $\square$

Question 6.3 [**10 points**] Prove your conjecture from Questions 6.2 formally. *The proof should be by induction on something. Prove base case and step case, and write out clearly the induction hypothesis.*

The proof is by induction on the hight of the tree.

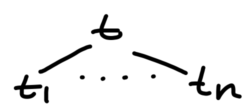
base case: Tree has hight 0 and has therefore only one leaf.

$n^0 = 1$ ✓

step case: Let k be the hight, arbitrary but fixed.

the n-ary tree has n subtrees. $t_1 \dots t_n$.

we can assume that for each subtree t_i the number of leaves are n^k.

to show: the number of leaves for  is $n^{(k+1)}$.

$$\#leaves(t) = \#leaves(t_1) + \dots + \#leaves(t_n) = \sum_{i=1}^{n} \#leaves(t_i) = \sum_{i=1}^{n} n^k = n \cdot n^k = n^{(k+1)}$$

$\square$