Exercises for Lecture 4: Induction

Let $\mathbb{N}^+$ be the set of all integers excluding zero; $\{1, 2, 3, \ldots\}$.

Exercise 1

To prove by induction that some $P(n)$ holds for every $n \in \mathbb{N}^+$, you need to prove two cases. What are these two cases and what do you need to prove in each of them?

Exercise 2

Let $P(n)$ be the following predicate:

$$1^3 + 2^3 + 3^3 + ... + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

1. What are the statements $P(1), P(2), P(3)$? Are they true?
2. Prove by induction that $\forall n \in \mathbb{N}^+.P(n)$.

Exercise 3

Let $P(n)$ be the following predicate: $n^2 + n + 41$ is a prime number.

1. Verify that $P(1), P(2), P(3)$ holds. (You can use this list of primes: http://en.wikipedia.org/wiki/List_of_prime_numbers)
2. Does your answer to (1) convince you that $P(n)$ holds $\forall n \in \mathbb{N}^+$?
3. Prove or disprove the statement $P(40)$. (Hint: Is 41^2 a prime number?)
4. Can you prove by induction that $\forall n \in \mathbb{N}^+.P(n)$? If yes, do it!
   If not, explain which step you won't be able to prove.

Exercise 4

Let $P(n)$ be the following predicate: 4 divides $5^n - 1$.
That an integer q divides an integer r is defined by $\exists p \in \mathbb{N}.r = pq$.

1. Express $P(n)$ according to the definition above.
2. Prove by induction that $\forall n \in \mathbb{N}^+.P(n)$.