SGDM E2011: Discrete Mathematics
Carsten Schürmann
Date: December 20, 2011

Final Mock Examination

Instructions

- This is a open book examination.
- There are 4 pages.
- This examination consists of 6 questions worth 100 points. The point value of each question is given with the question.
- Read each question completely before attempting to solve any part.
- Write your answers legibly.

Question	1	2	3	4	5	6	Total
Score							
Maximum	15	15	20	20	10	20	100

1 Logic [15 points]

A zookeeper is in charge for feeding the animals. In the zoo there are bears, giraffes, lions, monkeys and zebras. Here is his rule book:

1. The giraffes were fed before the zebras but after the monkeys.
2. The bears were fed after the monkeys.
3. The lions were fed after the zebras.

Your job is to determine the order in which the animals get fed in. We declare the following propositions: *bears(n)*, *giraffes(n)*, *lions(n)*, *monkeys(n)*, and *zebras(n)*.

For the position in order, we simply use $1, 2, 3, 4, 5$, and you can compare two positions by $<$, $=$, and $>$.

Questions 1.1 [5 points] Using first-order logic translate the three rules from above into logic.

Questions 1.2 [10 points] Are the giraffes being fed before the lions? Prove it formally, using the axioms, you have determined in 1.1.

2 Induction [15 points]

Question 2.1 [5 points] State the induction principle for induction over the natural numbers (starting at 1)?

Question 2.2 [10 points] Prove the following statement by induction on n

$$\text{For all } n > 1, \text{ it holds that } n^2 \geq 2n.$$

3 Complexity [20 points]

Consider the set of 7 functions:

$$n^{-3} \tag{1}$$
$$n^3 \tag{2}$$
$$\log n \tag{3}$$
$$3 \tag{4}$$
$$3^n \tag{5}$$
$$10^{6n} \tag{6}$$
$$n \log n \tag{7}$$

Question 3.1 [5 points] Recall the definition of big-O notation.

Question 3.2 [10 points] Order these functions, such that if $f(n) < g(n)$ if and only if and only if $f(n) = O(g(n))$. Prove that the order is correct. *Hint:* Give the c and k constants that you might have used in the definition of 3.1

Question 3.3 [5 points] Classify the following 3 functions according to their complexity class (in big O-notation).

1. $f(n) = 15n^3 + 12n^2 - 9n + 1$.
2. $g(n+1) = g(n) + n$, where $g(0) = 0$
3. $h(n+1) = h(n) + h(n)$ where $h(0) = 1$

4 Regular Expressions [20 points]

Recall from class the definition of regular expressions: $R ::= 1 \mid 0 \mid c \mid R_1 + R_2 \mid R_1 \times R_2 \mid R^*$.

Consider the language $\mathcal{L}$ of binary strings over $\{a, b\}$ that contains multiples of 3 b's. For example $\{abbab, abbbaaababab, bbab\} \subset \mathcal{L}$.

Question 4.1 [5 points] Write out at least three different regular expressions $R \neq 1$ whose language only contains the empty string.

Question 4.2 [5 points] Give a regular expression that generates the language $\mathcal{L}$.

Question 4.3 [5 points] Define a deterministic finite automaton that accepts $\mathcal{L}$. Define clearly the states. You may draw a diagram.

Questions 4.4 [5 points] Sketch a proof of why the automaton accepts $\mathcal{L}$. *Hint: Think of the property that remains invariant during for each of the states an automaton may be in.*

5 Models of computation [10 points]

In class we discussed the λ-calculus. Consider the following λ-terms

$$((\lambda x.\, \lambda y.\, x(y - 21))(\lambda z.\, 4 \cdot z))6$$

that can be thought of as a delegate in Java.

Question 5.1 [5 points] How many redexes can you find. Highlight them.

Question 5.2 [5 points] Give a reduction of this λ-expression to normal form.

6 Graphs [20 points]

Recall from class the definition of directed graphs and strongly connected components.

Question 6.1 [**5 points**] Give the definition of when a directed graph is *strongly connected*.

Question 6.2 [**5 points**] Give the definition of a *strongly connected component* of a graph.

Question 6.2 [**10 points**] Given a graph. Select a strongly connected component within the graph. Pick two vertices and a directed path in between the two vertices. Show that any vertex on that path is also a vertex in the same strongly connected component.