Lecture 2: Logic

In today's lecture, we will give a little introduction the philosophy and history of mathematical logic. Mathematical logic is not as old one one might think. While the first attempts to logical reasoning go back to Aristotle, it was only in the second half of the 19th century that mathematicians started thinking of logic as a field of scientific study. Until then logic was simply seen as the rules of the game that is called mathematics. There was no discussion about different logics and nobody really entertained the possibility that logical systems have properties and exhibit behaviors that are worth studying.

Nowadays mathematical logic as more important then ever. This is mostly because of computer science. Computers are very logical machines, we all might have heard about references to Boolean algebra and the like. During your stay at the IT University, you will hear a lot about good programming hygiene and contracts, a way to communicate important implicit information between a program/method that calls and the program/method that is being called. The language in which these contracts are expressed is logic. You will see that programming systems reason logically, which means that there is a *syntactic* side to logic that makes it more computer sciency, then its *semantic* side in the sense of classical mathematics.

> The tradition called syntactic for want of a nobler title never reached the level of its rival. In recent years, during which the algebraic tradition has flourished, the syntactic tradition was not of note and would without doubt have disappeared in one or two more decades, for want of any issue or methodology. The disaster was averted because of computer science that great manipulator of syntax which posed it some very important theoretical problems.
> *Jean-Yves Girard, Yves Lafont and Paul Taylor, 1990*

Because of this, we slightly deviate from the standard way of treating logic in a discrete mathematics class, and start with a, in my opinion particularly elegant way to combine logic, mathematics, and computing. We will use this logical system for everything throughout this discrete math class: *Reasoning*, *programming*, *specifying*, and even *discovering* flaws.

The central concept of our view of logic is that of a *judgment*. Judgments are for example that A is a formula and perhaps most importantly *the formula A is true*, which we shall abbreviate as $A\ \mathsf{true}$. In this judgment A stands for any mathematical formula. Theses formulas define the language of mathematics that we will be using extensively in this course. We say that A is true if it $A\ \mathsf{true}$ can be derived with a systems of inference rules. Both, the languages of formulas, and the rules of inference are going to be described in the remainder of this section. The overall goal of this lecture is therefore to introduce the common language and to start internalizing by many examples.

Let's start for real. We must now introduce each one of the connectives defining the logic that we will be using throughout this entire class. First, I would like to remark, that our logical presentation is completely void of any domain: Instead, we stipulate a family of propositional formulas, called P, that could stand for things like state(white, white, white), from last weeks lecture, even(n), odd(n), or even prime(n). Yes, we will need to fill logic with mathematical life, but not today. Today we will work on understanding what mathematical reasoning is all about, according to which rules are we allowed to reason and how do we actually do it.

Conjunction The first connective, we tackle is conjunction. The usual "and". If A and B are two formulas, then $A \wedge B$ is a formula. Next we define the respective inference rules that allow us to reason with a formula. i.e. a rule that introduces the conjunction $A \wedge B$ true, and two rules that eliminate $A \wedge B$ true again. Don't be scared, just bare with me.

$$\frac{A\ \mathsf{true} \quad B\ \mathsf{true}}{A \wedge B\ \mathsf{true}}\wedge\mathrm{I} \qquad \frac{A \wedge B\ \mathsf{true}}{A\ \mathsf{true}}\wedge\mathrm{E}_1 \qquad \frac{A \wedge B\ \mathsf{true}}{B\ \mathsf{true}}\wedge\mathrm{E}_2$$

Let p be a proposition stating that the ball is red. and q be a proposition stating that the grass is green. Then we can infer that using the $\wedge$I rule, that the ball is red and the grass is green from the facts fact1 that the ball is really red and fact2 that the grass is green:

$$\frac{\dfrac{}{p\ \mathsf{true}}\mathrm{fact1} \qquad \dfrac{}{q\ \mathsf{true}}\mathrm{fact2}}{p \wedge q\ \mathsf{true}}\wedge\mathrm{I}$$

Very often, we find this way of writing proofs annoyingly cumbersome, and describe the individual proof steps as a list. We also omit the true from the judgment, to make things more readable.

Lemma 8 *Let fact1 and fact2 as above. Then we can prove $p \wedge q$.*

Proof:

p	by fact1
q	by fact2
$p \wedge q$	by $\wedge$I

□

This is our first formal proof. When a proof is done, we usually mark its end with a little box □, or we write *q.e.d.*, which stands for quod erat demonstrandum. With conjunction alone, however, we cannot prove many interesting things. We need to add a bit more power to get things off the ground.

Implication Let's do it and introduce implication, which is a way to reason hypothetically. If I want to study the question if p implies q, then we may assume that we have a proof of p to proof q. Therefore, the meaning of implication is best described by the following two rules.

$$\dfrac{\begin{array}{c}\overline{A\ \mathsf{true}}\,u\\ \vdots\\ B\ \mathsf{true}\end{array}}{A \Rightarrow B\ \mathsf{true}} \Rightarrow \mathrm{I}^u \qquad \dfrac{A \Rightarrow B\ \mathsf{true} \quad A\ \mathsf{true}}{B\ \mathsf{true}} \Rightarrow \mathrm{E}$$

The $\vdots$ stands for a derivation, that may use the additional assumption that A true. Let's do a small example. Can you prove that if $p \Rightarrow (q \Rightarrow p)$? Let's do it together.

$$\dfrac{\dfrac{\overline{p\ \mathsf{true}}\,u}{(q \Rightarrow p)\ \mathsf{true}} \Rightarrow \mathrm{I}^v}{p \Rightarrow (q \Rightarrow p)\ \mathsf{true}} \Rightarrow \mathrm{I}^u$$

Success. Let's formulate this fact as a mathematical lemma and give a linearized version of the proof. A proof that a mathematician would write out.

Lemma 9 *Let p and q be arbitrary predicates (even formulas, if one wants): Then we can prove that $p \Rightarrow (q \Rightarrow p)$.*

Assume p (called u above)	
Assume q (called v above)	
p	by assumption
$q \Rightarrow p$	by $\Rightarrow$ I discharging assumption v
$p \Rightarrow (q \Rightarrow p)$	by $\Rightarrow$ I discharging assumption u

□

For the second example, I would like to return to last weeks lecture, and the example of the flipping pebble game. Let's limit our attention to 3 pebble games only.

Recall the specification:

- If the current pebble is white, color it blue and skip the next two.
- If the current pebble is blue, color it white and skip the next.

Let $state(x, y, z)$ be a predicate (indexed by three arguments that correspond to pebble 1, 2, and 3). The specification can be expressed by the following eight rules.

1. state(white, white, white) $\Rightarrow$ state(blue, white, white).

2. state(white, white, blue) $\Rightarrow$ state(blue, white, blue).
3. state(white, blue, white) $\Rightarrow$ state(blue, blue, white).
4. state(white, blue, blue) $\Rightarrow$ state(blue, blue, blue).
5. state(blue, white, white) $\Rightarrow$ state(white, white, blue)
6. state(blue, blue, white) $\Rightarrow$ state(white, blue, blue)
7. state(blue, white, blue) $\Rightarrow$ state(white, white, white)
8. state(blue, blue, blue) $\Rightarrow$ state(white, blue, white).

Theorem 10 *We can find a non-trivial derivation of state(white, white, white) $\Rightarrow$ state(white, white, white), i.e. a derivation that doesn't just use the assumption but requires some reasoning with rules.*

Proof:

Assume state(white, white, white) (called u)	
state(blue, white, white)	by $\Rightarrow$ E using (1).
state(white, white, blue)	by $\Rightarrow$ E using (5).
state(blue, white, blue)	by $\Rightarrow$ E using (2).
state(white, white, white)	by $\Rightarrow$ E using (7).
state(white, white, white) $\Rightarrow$ state(white, white, white)	by $\Rightarrow$ I discharging u.

$\square$

Truth and falsehood No logic is complete with truth and falsehood. From your high school math class, you might remember that we put true and false, 1 and 0 into the center of mathematics. A theorem is either true or false. My main message for this lecture is that in order to do discrete math for computer science and information technology, we should not be only interested if we can write a certain program or not, we actually must be write the program and it better runs well according to specification.

The meaning of truth and falsehood is specified by the following two rules. We can always derive true, and if we can derive false then we can derive whatever we want (C true).

$$\frac{}{\top\ \mathsf{true}}\,\top\mathrm{I} \qquad \frac{\bot\ \mathsf{true}}{C\ \mathsf{true}}\,\top\mathrm{E}$$

Negation Closely related to falsehood is negation. Here is how the two are related. We write $\neg A$ for the negation of A, and define it as follows. If A true then C true for an arbitrary C. This immediately justifies the introduction rule for negation:

$$\dfrac{\dfrac{}{A\ \mathsf{true}}u \\ \vdots \\ p\ \mathsf{true}}{\neg A\ \mathsf{true}}\neg\mathrm{I}^{u,p}$$

p is a parameter that ranges over formulas. The elimination rule is as follows.

$$\dfrac{\neg A\ \mathsf{true} \quad A\ \mathsf{true}}{C\ \mathsf{true}}\neg\mathrm{E}$$

These are the reasoning rule that underlying every single proof that we will be discussing in this course. We will have to gain some experience from them, and internalize them.

Theorem 11 (Modus Tollens) *If* $A \Rightarrow B$ true *and* $\neg B$ true *then* $\neg A$ true.

Proof:

Assume A true (called u)
Assume p arbitrary formula

$A \Rightarrow B$ true	by assumption
B true	by $\Rightarrow$ E
$\neg B$ true	by assumption
p true	by $\neg$E
$\neg A$ true	by $\neg$I discharging u and p

□

Disjunction Let's look at one other connective, disjunction, for which we write $A \vee B$. Its meaning is defined by two introduction rules

$$\dfrac{A}{A \vee B}\vee\mathrm{I}_1 \qquad \dfrac{B}{A \vee B}\vee\mathrm{I}_2$$

and one eliminiation rule.

$$\dfrac{A \vee B \quad \begin{matrix}\overline{A}^{\,u} \\ \vdots \\ C\end{matrix} \quad \begin{matrix}\overline{B}^{\,v} \\ \vdots \\ C\end{matrix}}{C}\vee\mathrm{E}^{u,v}$$

This is the base system. It's simple, beautiful and clean. But not all of mathematics can be done in it. Mathematicians tend to add axioms to logic.

The rule of the excluded middle is such an axiom. When we do this, we have classical logic, but it is difficult to attribute a computational meaning to the law of the excluded middle.

$$\frac{}{A \vee \neg A \ \mathsf{true}}\ exm$$

This axiom is not derivable in intuitionistic logic. We can try:

Attempt 1:

A	????
$A \vee \neg A$	by $\vee I_1$

Attempt 2:

Assume A true	
Assume p, an arbitrary formula	
p true	????
$\neg A$	by $\neg$I
$A \vee \neg A$	by $\vee I_2$

Our logic (also called constructive logic, or intuitionistic logic), together with this axiom gives us classical logic, a logic that mathematicians usually work in. In classical logic we can prove theorems that we cannot proof in intuitionistic logic (simply because we assume more by the law of the excluded middle). If you are interested in this, you need to take a logic class.

The law of double negation introduction is derivable only in classical logic,

$$\frac{\neg\neg A}{A}\ dnE$$

double negation elimination is derivable in classical logic, and constructive logic as well.

$$\frac{A}{\neg\neg A}\ dnI$$

Theorem 12 *If* A true *then* $\neg\neg A$ true.

Proof:

Assume $\neg A$ true	
Assume p an arbitrary formula	
A true	by assumption
p true	by rule $\neg$E
$\neg\neg A$ true	by rule $\neg$I

□

Theorem 13 *If we assume the law of the excluded middle then it holds that if* $\neg\neg A$ true *then* A true*.*

Proof:

$\neg\neg A$ true	by assumption
$A \vee \neg A$ true	by rule *exm*
Assume A true	
A true	by assumption
Assume $\neg A$ true	
A true	by $\neg$E
A true	by $\vee$E.

□

Theorem 14 *If we assume that double negation introduction holdes, then* $A \vee \neg A$ *is derivable for all formulas* A*.*

Proof:

Assume $\neg(A \vee \neg A)$ true	
Assume p an arbitrary formula	
Assume A true	
Assume q an arbitrary formula	
$A \vee \neg A$	by $\vee I_1$
q true	by $\neg$E
$\neg A$	by $\neg$I
$A \vee \neg A$	by $\vee I_2$
p true	by $\neg$E
$\neg\neg(A \vee \neg A)$ true	by $\neg$I
$A \vee \neg A$ true	by DN.

□

In constructive logic, a formula is not just true or false, it either has a constructive proof or not. In classical logic, provability is not the central concern, it is validity, and therefore the meaning of a formulas is aways true or false. Everything that you have learned in a high school or undergraduate mathematics is usually all classical. We are constructive. The good news is that many of the theorems that are classically valid also have proofs in intuitionistic logic. One just has to work a bit harder.

This concludes our second lecture. We have encountered all the usual connectives, and have explained their meaning in form of inference rules. We have discussed truth and falsehood, conjunction, negation, disjunction, implication, and we have encountered the rules for the excluded middle, and double negation introduction and elimination.