Lecture 4: Induction

Since Kindergarten, we all know the natural numbers and how to generate them. We start with 0 as a natural number, if n is a natural number then $n+1$ is also a natural number. It is pretty easy to see, that any number can we written as a large sum:

$$n = 0 + \underbrace{1 + \ldots + 1}_{n \text{ times}}$$

We say that the numbers are inductively defined, by two constructors, 0 and +1.

The induction principle differs significantly from the introduction rule above. It is designed to analyze the structure of a term.

Let's consider the situation that we need to show something for all numbers: $\forall n \in I\!N. P(n)$. Since the rule for universal quantification introduction does not allow us to inspect the structure of a general n, theorems of that kind are not always directly provable. Very often the proof only goes through when we look at the structure of n.

Since we know that all numbers are constructed from these two constructors, we could try to prove P for every concreate instantiation of n such as

$$P(0), P(1), P(2), P(3), \ldots$$

It is pretty clear that this would take forever, and therefore not result in a proof. This is not ideal. Here is another idea. We should try to justify $P(n+1)$ based on the knowledge that $P(n)$ holds. This means we have to prove two cases, first the base case,

$$P(0)$$

and secondly the step case

$$\forall n \in I\!N. P(n) \to P(n+1)$$

If we provide a proof of these two cases, then we have conducted a proof by induction on the structure of that natural number n. In the case that we cannot prove the second case directly (using our rules from last lecture + the standard unification introduction and elimination rules), we might have to do a second nested induction on n again.

Once these two cases are proved, we can apply the theorems and convince ourselves that, if we only had infinite amounts of time, we could convince ourselves that $P(n)$ is true for any $n \in I\!N$.

$P(0)$	from the first case
$P(0) \to P(1)$	form the second case with $n = 0$
$P(1)$	by $\to$ E
$P(1) \to P(2)$	form the second case with $n = 1$
$P(2)$	by $\to$ E
...	

All we have to believe is that this argument scales to the infinite. What could go wrong? Nothing, but we have to admit, this proof constructions is somewhat less intuitive then the other. We must believe that everything works out ok in the infinite. It is not justified/justifiable in any other way then what I just showed you. Note, that very few mathematicans distrust this prinicple.

This completes the technical part of this lecture. The cool thing is that using the principle of induction, we can actually prove quite intersting things, about numbers. Here is one, a theorem that is named after Carl Friedrich Gauss "the little Gauss". When he was eight years old, Gauss's class was asked to add all numbers between 1 and 100. Gauss answered immediately 5050. The teacher asked him how he did it: Add the first number and the last, $1+100 = 101$. Then you add $2+99 = 101$, then $3+98 = 101$, etc. until you come to $50+51 = 101$. Hence you have $50 \times 101 = 550$.

Let's generalize it a little bit. What happenes if we do not only want to add up the first 100 numbers, but the first n numbers? We introduce some notation, which will make it much easier for us to formulate the theorem. We write

$$\sum_{i=0}^{n} i$$

as an abbreviation for $0 + 1 + 2 \ldots n$.

Theorem 20

$$\forall n \in I\!N. \sum_{i=0}^{n} i = \frac{n(n+1)}{2}$$

Proof: How do we prove it? Of course by induction over n. We must consider two case.

Case 1

$\sum_{i=0}^{0} i = 0 = \frac{0(0+1)}{2}$ by equational reasoning

Case 2

Let n be arbitray but fixed
Assume $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$ called u
$\sum_{i=0}^{n+1} i$
$= \sum_{i=0}^{n} i + (n+1)$ by the definition of $\sum$
$= \frac{n(n+1)}{2} + (n+1)$ be replacing equals for equals
$= (\frac{n}{2} + 1)(n+1)$ by pulling out the $(n+1)$
$= (\frac{n+2}{2})(n+1)$
$= (\frac{(n+1)+1}{2})(n+1)$
$= \frac{((n+1)+1)(n+1)}{2}$

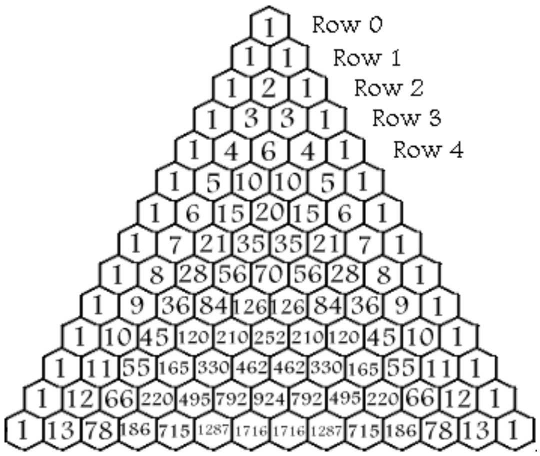

Figure 1: Pascal's triangle

$= \frac{(n+1)((n+1)+1)}{2}$

$\sum_{i=0}^{n} i = \frac{n(n+1)}{2} \rightarrow \sum_{i=0}^{(n+1)} i = \frac{(n+1)((n+1)+1)}{2}$ by $\rightarrow$ I discharging u

$\forall n \in \mathbb{N}. \sum_{i=0}^{n} i = \frac{n(n+1)}{2} \rightarrow \sum_{i=0}^{(n+1)} i = \frac{(n+1)((n+1)+1)}{2}$ by $\forall$I discharging n

$\forall n \in \mathbb{N}. \sum_{i=0}^{n} i = \frac{n(n+1)}{2}$ by the principle of induction

□

The second object that we will study today is Pascal's triangle. Consider a board with a rows of nail. The top row has one nail, the second row two, the third three and so forth. What happens if you drop a ball on the top nail. It can go either left or right. If it goes left it hits the first nail of the second column, and again, it can go either left or right, etc. You can actually run these experiments in the physical world, there are plenty of clips on youtube demonstrating this.

When you let a ball fall through the nails, where is going to end up? If we run the experiment more than once, we recognize a pattern. That a ball ends up in the center is much more likely than it ending up close to the sides. But why? How can we understand it.

For every point right under a nail, let us count how many paths there are. If we do this carefully, we obtain the triangle whose first 14 rows are depicted in Figure 1.

Pascal's triangle is very famous. It exhibts a lot of interesting patterns with respect to its numbers. For example, every row seems to add up to 2^n, where n is the row number. There are many ways to compute an entry in Pascal's triangle. One is

$$P(n,k) = \begin{cases} 1 & \text{if } n = 0 \wedge k = 0 \\ P(n-1,k-1) + P(n-1,k) & \text{if } n \geq 0 \wedge k \geq 0 \wedge n \geq k \\ 0 & \text{otherwise} \end{cases}$$

Let's check, if this is right:

$$\begin{aligned} P(3,2) &= P(2,1) + P(2,2) \\ &= P(1,0) + P(1,1) + P(1,1) + (1,2) \\ &= P(0,-1) + P(0,0) + P(0,0) + P(0,1) + P(0,0) + P(0,1) + P(0,1) + P(0,2) \\ &= 0 + 1 + 1 + 0 + 1 + 0 + 0 + 0 \\ &= 3 \end{aligned}$$

If we have the definition of the factorial function.

$$n! = \begin{cases} 1 & \text{if } n < 0 \\ n \times (n-1)! & \text{otherwise} \end{cases}$$

We can try to prove that

Theorem 21 *Let* $0 \leq k \leq n$

$$P(n,k) = \frac{n!}{k!(n-k)!}$$

Proof: by induction on n

Case: $n = 0$. Thus $k = 0$. $1 = P(0,0) = \frac{0!}{0! * 0!} = 1$

Case: Let n' be arbitrary but fixed. Let $n = n' + 1$. Assume that for an arbitrary k' the following holds

$$P(n',k') = \frac{n'!}{k'!(n'-k')!}$$

Subcase: $k' = 0$:

$$\begin{aligned} P(n'+1,0) &= P(n',-1) + P(n',0) = P(n',0) = \frac{n'!}{n'!} = 1 \\ &= \frac{(n'+1)!}{0!(n'+1-0)!} \end{aligned}$$

Subcase: $k' = n' + 1$:

$$\begin{aligned} P(n'+1,n'+1) &= P(n',k') + P(n',n'+1) = P(n',n') = \frac{n'!}{n'!} = 1 \\ &= \frac{(n'+1)!}{(n'+1)!(n'+1-(n'+1))!} \end{aligned}$$

Subcase: $0 < k' < n' + 1$:

$$
\begin{aligned}
P(n'+1, k') &= P(n', k'-1) + P(n', k') \\
&= \frac{n'!}{(k'-1)!(n'-k'+1)!} + \frac{n'!}{k'!(n'-k')!} \\
&= \frac{n'!k'!(n'-k')! + n'!(k'-1)!(n'-k'+1)!}{(k'-1)!(n'-k'+1)!k'!(n'-k')!} \\
&= \frac{n'!k'!(n'-k')! + n'!(k'-1)!(n'-k')!(n-k'+1)}{(k'-1)!(n'-k'+1)!k'!(n'-k')!} \\
&= \frac{n'!k'! + n'!(k'-1)!(n'-k'+1)}{(k'-1)!(n'-k'+1)!k'!} \\
&= \frac{n'!(k'-1)!k' + n'!(k'-1)!(n'-k'+1)}{(k'-1)!(n'-k'+1)!k'!} \\
&= \frac{n'!k' + n'!(n'-k'+1)}{(n'-k'+1)!k'!} \\
&= \frac{n'!(k' + (n'-k'+1))}{(n'-k'+1)!k'!} \\
&= \frac{n'!(n'+1)}{(n'-k'+1)!k'!} \\
&= \frac{(n'+1)!}{(n'-k'+1)!k'!} \\
&= \frac{(n'+1)!}{k'!((n'+1)-k')!}
\end{aligned}
$$

□