SGDM E2012: Discrete Mathematics
Carsten Schürmann
Date: January 4, 2013

Final Examination

Instructions

- This is a closed book examination.
- The examination consists of 4 pages.
- This examination consists of 7 questions worth 100 points. The point value of each question is given with the question.
- Read each question completely before attempting to solve any part.
- Write your answers legibly.

1 Logic [15 points]

Recall from class the following rules defining conjunction, implication, disjunction, and negation.

$$\frac{A \quad B}{A \wedge B}\wedge\mathrm{I} \qquad \frac{A \wedge B}{A}\wedge\mathrm{E}_1 \qquad \frac{A \wedge B}{B}\wedge\mathrm{E}_2$$

$$\frac{\begin{array}{c}\overline{A}^{\,u}\\ \vdots \\ B\end{array}}{A \Rightarrow B}\Rightarrow\mathrm{I}^u \qquad \frac{A \Rightarrow B \quad A}{B}\Rightarrow\mathrm{E}$$

$$\frac{A}{A \vee B}\vee\mathrm{I}_1 \qquad \frac{B}{A \vee B}\vee\mathrm{I}_2 \qquad \frac{A \vee B \quad \begin{array}{c}\overline{A}^{\,u}\\ \vdots \\ C\end{array} \quad \begin{array}{c}\overline{B}^{\,v}\\ \vdots \\ C\end{array}}{C}\vee\mathrm{E}^{u,v}$$

$$\frac{\begin{array}{c}\overline{A}^{\,u}\\ \vdots \\ \perp\end{array}}{\neg A}\neg\mathrm{I}^u \qquad \frac{\neg A \quad A}{C}\neg\mathrm{E}$$

Questions 1.1 [5 points] Give a derivation of

$$(A \wedge B) \Rightarrow (A \vee B)$$

Questions 1.2 [10 points] Give a derivation of

$$(\neg A \wedge \neg B) \Rightarrow ((A \vee B) \Rightarrow C)$$

2 Induction [15 points]

Consider the Fibonacci numbers $fib(n)$ that we already defined in class for all $n \geq 0$.

$$fib(n) = \begin{cases} 1 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ fib(n-1) + fib(n-2) & \text{if } n \geq 2 \end{cases}$$

Question 2.1 [5 points] Let's get a feel for this function. Compute $fib(7)$.

Question 2.2 [10 points] Use mathematical induction to prove that for all $n \geq 0$

$$fib(n) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

Hint: Prove base case and step case, and write out clearly the induction hypothesis. You might want to use that $(x+y)^2 = x^2 + 2xy + y^2$ *and* $(x-y)^2 = x^2 - 2xy + y^2$.

3 Complexity [10 points]

Consider the function

$$f(n) = 4n^6 - 19n^3$$

You might think of this function as the number of steps necessary to compute some output from an input of size n.

Question 3.1 [5 points] Show that $f(n) \in O(n^6)$.

Question 3.1 [5 points] Let f be defined as above and $g(n) = O(n^3)$. Show that $f(n) \times g(n) = O(n^9)$.

4 Regular Expressions [15 points]

Recall from class the definition of regular expressions: $R ::= 1 \mid 0 \mid c \mid R_1 + R_2 \mid R_1 \times R_2 \mid R^*$. A *string* is a word, a sequence of characters. The c denotes a character from the alphabet.

Question 4.1 [5 points] Write out a regular expression that accepts all strings over the alphabet $\{a, b\}$.

Question 4.2 [5 points] Consider the language $\mathcal{L}$ of strings over the alphabet $\{a, b\}$ that contains an odd number of bs and at least one a. For example the strings $abaaabb$, ab, or aba are contained in the language $\mathcal{L}$. The strings $abba$ and $aabab$ are not. Give a regular expression that generates the language $\mathcal{L}$.

Question 4.3 [5 points] Draw a deterministic finite automaton that accepts $\mathcal{L}$. Define clearly the states. Draw a diagram.

5 Models of Computation [15 points]

Consider the following λ-term

$$(\lambda f.\, f\ (\lambda x.\, x+3))\ (\lambda g.\, g\ (g\ 4)).$$

Question 5.1 [5 points] How many redices can you find. Highlight them.

Question 5.2 [10 points] Give a reduction of this λ-expression to normal form.

6 Trees [20 points]

Recall from class the definition of a *binary* tree, where every node has exactly 2 children or is a leaf. Values are stored in the nodes and in the leaves. The *search tree invariant* states that all values stored in the *left* subtree of a node are strictly less than the value at the node, and all the values stored in the *right* subtree are strictly greater.

Question 6.1 [5 points] Give a binary search tree of height 3 that satisfies the search invariant and contains the following values: $4, 17, 25, 2, 16, 3, 21, 15, 9, 12, 1, 22, 7, 11, 19$.

Question 6.2 [5 points] How many elements does a binary search tree of height n need to contain, in order to be unique? Explain your solution.

Question 6.3 [10 points] Give a counter example of a binary search tree of height 2 that has one fewer elements than you claim in 6.2.

7 Probability Theory [10 points]

Suppose that A and B are events in a sample space with $P(A) = 0.4$, $P(B) = 0.25$, and $P(A|B) = 0.3)$. Let us denote the complement of B as $\overline{B}$. What is $P(A|\overline{B})$? Explain your solution.