Logical- and Meta-Logical Frameworks
Lecture 4

Carsten Schürmann

August 10, 2006

Recall from last time

Conclusion

- Twelf is meta logical framework.
- Logic Programming Semantics give raise to new function arrow.
- Totality = Modes + World + Termination + Coverage.

Homework

- Prove that if n is odd and m is even, then $n + m$ is odd.
- Extra: Implement the double negation theorem.

Today we do a larger example:
The cut-elimination for first-order logic.

Logic

Formulas	$A, B, C ::= p \mid \top \mid \bot \mid \neg A \mid A \supset B \mid \forall x.\, A$
Terms	$T ::= a$
Assumptions	$\Delta ::= \cdot \mid \Delta, A \mid \Delta; p \mid \Delta; a$
Comments	

- No predicates symbols (no equality)
- No term symbols

Sequent calculus

Judgment $\Delta \Longrightarrow A$

Rules

$$\frac{}{A \Longrightarrow A}\text{axiom} \qquad \frac{}{\Longrightarrow \top}\text{topR} \qquad \frac{}{\bot \Longrightarrow C}\text{botL}$$

$$\frac{\Delta \Longrightarrow A}{\Delta, \neg A \Longrightarrow C}\text{negL} \qquad \frac{\Delta; p, A \Longrightarrow p}{\Delta \Longrightarrow \neg A}\text{negR}$$

$$\frac{\Delta \Longrightarrow A \quad \Delta, B \Longrightarrow C}{\Delta, A \supset B \Longrightarrow C}\text{impL} \qquad \frac{\Delta, A \Longrightarrow B}{\Delta \Longrightarrow A \supset B}\text{impR}$$

$$\frac{\Delta, [T/x]A \Longrightarrow C}{\Delta, \forall x.\, A \Longrightarrow C}\text{allL} \qquad \frac{\Delta; a \Longrightarrow [a/x]A}{\Delta \Longrightarrow \forall x.\, A}\text{allR}$$

Structural rules

$$\frac{\Delta, A, A \Longrightarrow C}{\Delta, A \Longrightarrow C}\ \text{contraction}$$

$$\frac{\Delta \Longrightarrow C}{\Delta, A \Longrightarrow C}\ \text{weakening}$$

$$\frac{\Delta, A, B \Longrightarrow C}{\Delta, B, A \Longrightarrow C}\ \text{exchange}$$

Observation

- Drop contraction: Affine Logic
- Drop weakening: Relevant Logic
- Drop weakening and contraction: Linear Logic
- Drop all: Lambek calculus

Sequent calculus (incl structural rules)

Judgment $\Delta \Longrightarrow A$

Rules

$$\frac{}{\Delta, A \Longrightarrow A}\text{axiom} \qquad \frac{}{\Delta \Longrightarrow \top}\text{topR} \qquad \frac{}{\Delta, \bot \Longrightarrow C}\text{botL}$$

$$\frac{\Delta, \neg A \Longrightarrow A}{\Delta, \neg A \Longrightarrow C}\text{negL} \qquad \frac{\Delta; p, A \Longrightarrow p}{\Delta \Longrightarrow \neg A}\text{negR}$$

$$\frac{\Delta, A \supset B \Longrightarrow A \qquad \Delta, A \supset B, B \Longrightarrow C}{\Delta, A \supset B \Longrightarrow C}\text{impL} \qquad \frac{\Delta, A \Longrightarrow B}{\Delta \Longrightarrow A \supset B}\text{impR}$$

$$\frac{\Delta, \forall x.\, A, [T/x]A \Longrightarrow C}{\Delta, \forall x.\, A \Longrightarrow C}\text{allL} \qquad \frac{\Delta; a \Longrightarrow [a/x]A}{\Delta \Longrightarrow \forall x.\, A}\text{allR}$$

Representation in LF

Derivations

$$\ulcorner \mathcal{D} :: \Delta \Longrightarrow A \urcorner \quad = \quad \ulcorner \Delta \urcorner \vdash \ulcorner D \urcorner : \mathsf{conc}\ulcorner A \urcorner$$

Contexts

$$\begin{array}{rcl} \ulcorner \cdot \urcorner & = & \cdot \\ \ulcorner \Delta, A \urcorner & = & \ulcorner \Delta \urcorner, u : \mathsf{hyp}\ulcorner A \urcorner \\ \ulcorner \Delta, p \urcorner & = & \ulcorner \Delta \urcorner, p : \mathsf{o} \\ \ulcorner \Delta, a \urcorner & = & \ulcorner \Delta \urcorner, a : \mathsf{i} \end{array}$$

World checking

Blocks Defined the units of information contained in a context.

Block of *some variables* are existential quantified.

Block of *block variables* are universal.

```
%block l1 : some A:o block h:hyp A.
%block l2 : block a:i.
%block l3 : block p:o.
```

Worlds Check that an operational proogram never puts anything else but those blocks into the context.

```
%worlds (l1 | l2 | l3) (hyp A) (conc A).
```

Theorems

Theorem In the world described by `l1 | l2 | l3`.
If $\mathcal{D} :: \Delta \Longrightarrow A$ and $\mathcal{E} :: \Delta, A \Longrightarrow B$ then $\Delta \Longrightarrow B$.

Define $\Delta \Longrightarrow^* A$ all rules for $\Delta \Longrightarrow A$ +

$$\frac{\Delta \Longrightarrow^* A \qquad \Delta, A \Longrightarrow B}{\Delta \Longrightarrow^* B}\,\mathsf{cut}$$

Theorem In the world described by `l1 | l2 | l3`.
If $\Delta \Longrightarrow^* A$ then $\Delta \Longrightarrow A$.

Corollary First order logic is sound.

Proof

Schema: Induction on A, $\mathcal{D}$ and $\mathcal{E}$.

Axiom conversions, essential cases, commutative cases.

Proof: Blackboard

Representation: Theorems.

Representation: Proofs.

Conclusion

- First order logic, fits wonderfully into the framework.
- Importance of world checking.
- Proof by structural induction by cut elimination.
- Relies on coverage checking.
- Next time we will see how to conduct proofs by logical relations.